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THE HEBREW UNIVERSITY OF JERUSALEM

ON THE STRATEGIC USE OF SELLER INFORMATION IN PRIVATE-VALUE FIRST-PRICE AUCTIONS

By

TODD R. KAPLAN AND SHMUEL ZAMIR

Discussion Paper # 745 (December 2021)

מרכז פדרמן לחקר הרציונליות

THE FEDERMANN CENTER FOR THE STUDY OF RATIONALITY

Feldman Building, Edmond J. Safra Campus, Jerusalem 91904, Israel
PHONE: [972]-2-6584135 FAX: [972]-2-6513681
E-MAIL: ratio@math.huji.ac.il
URL: http://www.ratio.huji.ac.il/

On the Strategic Use of Seller Information in Private-Value First-Price Auctions*

Todd R. Kaplan[†] Shmuel Zamir[‡]

December 14, 2021

Abstract

In the framework of a private-value auction first-price, we consider the seller as a player in a game with the buyers in which he has private information about their realized values. We ask whether the seller can benefit by using his private information strategically. We find that in fact, depending upon his information, set of signals, and commitment power the seller may indeed increase his revenue by strategic transmission of his information. We study mainly the case of *partial truthful commitment* (VC) in which the seller can commit to send only truthful (verifiable) messages. We show that in the case of two buyers with values distributed independently uniformly on $[0,1]$, a seller informed of the private values of the buyers, can achieve a revenue close to $1/2$ by sending verifiable messages (compared to $1/3$ in the standard auction), and this is the largest revenue that he can reach with any signaling strategy and any level of commitment. The case studied here provides valuable insight into the issue of strategic use of information which applies more generally.

1 Introduction

In the framework of a first-price private-value auction, we ask whether a seller that has information about buyers' private values can strategically use his information so as to increase revenue. We see a hint of this possibility in Landsberger, Rubinstein, Wolfstetter, and Zamir (2001), who show in the standard first-price sealed-bid auction when the ranking of the values is announced and becomes common knowledge, revenue increases. For instance, when values are drawn independently from the uniform distribution on $[0,1]$, when ranking is common knowledge, *both* buyers bid more aggressively and consequently the revenue is 0.3696, which is larger than the revenue equal to $1/3$ in the standard model without the information about the ranking (see Figure 1).[1]

*There is an earlier working paper (WP 221) with the same title from 2000. This current paper has substantially evolved since then.

[†]Department of Economics, University of Exeter, EX44PU, UK, and Department of Economics, University of Haifa, Mount Carmel, Haifa, 31905, Israel. (Corresponding author)

[‡]The Federmann Center for the Study of Rationality, Hebrew University, Jerusalem, 91904, Israel.

[1]Martínez-Pardina (2006) finds a similar revenue increase if the valuation of the highest-value bidder is announced.

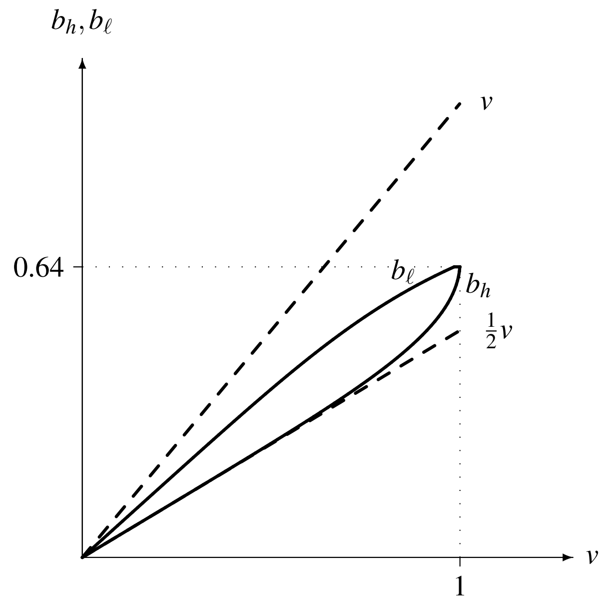

FIGURE 1. *When the ranking is common knowledge, revenue equals* $0.3696 > \frac{1}{3}$. *The lines* b_ℓ *and* b_h *are the bid functions of the low-value and high-value buyers, respectively.*

Why do we want to study this question? A seller (e.g., an auction house) may indeed have private information about the buyers' types from past auctions. It could also be information about the objects' attributes, which may affect in different ways the buyers' values. Finally, if information proves valuable, it may be worthwhile for the seller to try to acquire it.

There are two possible frameworks to study this question. The first is the mechanism design approach according to which, given his information, the seller chooses the optimal selling mechanism to maximize his revenue. The second is within a specific auction design. The seller may be restricted to a specific design because it may be impossible, illegal, or simply too costly to change the selling mechanism according to the available information.

Hence, in this paper, we take the latter approach and study the problem within the framework of the first-price auction. To do so, we consider the game in which the seller is a player whose strategy set is determined by the information he has, his message space, and his *commitment power* which turns out to play a very important role. For example, in Kaplan and Zamir (2000), we show that without commitment power of some form, the seller's information about the ranking is useless. This holds for any set of available signals, private and/or public. The intuition is that in Landsberger, Rubinstein, Wolfstetter, and Zamir (2001), the low ranked buyer is more aggressive than the high-ranked buyer. This gives the seller incentive to convince the high-ranked buyer that he is ranked lower and thus any such messages will be ignored due to this incentive to lie. On the other hand, if the seller can commit to send only *truthful* messages, then announcing the ranking makes the true ranking common knowledge and the induced game would be that of

Landsberger et al. with revenue equal to 0.3696 for the seller.[2]

There are three natural levels of commitment power. The first is *no commitment* power where the seller can say anything and it doesn't need to rely on his information. The other extreme is *full commitment* power in which the seller can commit to a signaling strategy (that is, he can submit a strategy to be executed by a neutral authority). In between, the seller may be restricted to messages that are *verifiable*, that is, the seller is limited to telling the truth (not necessarily to reveal all his information), but cannot commit to a specific signaling strategy. This level of commitment which we call *Partial Truthful Commitment* (which we shall also call *verifiability commitment*) will be the the focus of this work. Our example above demonstrates the importance of the commitment power. A seller, knowing the ranking of the values of the two buyers, cannot use this information to increase his revenue unless he has at least partial truthful commitment power.

We look at the standard first-price sealed-bid auction with two buyers with independent private values uniformly distributed in $[0,1]$. The seller knows the realized values (v_1, v_2) and sends public truthful (verifiable) messages prior to the bidding. We study the profits of the buyers and the revenue of the seller in the Bayesian-Nash equilibrium of this game. These depend on the set of signaling strategies available to the seller. We first establish that $1/2$ (which is $3/4$ of the *social surplus* of $2/3$) is the upper bound of the seller revenue compared to the revenue equal to $1/3$ ($1/2$ of the social surplus) without this information. Furthermore, this upper bound is valid for a seller with any set of signaling strategies and any commitment power.

It follows from Bergemann, Brooks, and Morris (2017), using a different setting, that this bound on revenue equal to $1/2$ can be achieved by a seller with full commitment power. We then ask: What can be achieved by a seller with partial truthful commitment power? We prove that this bound of $1/2$ can be achieved asymptotically with partial truthful commitment, that is, a seller can get a revenue arbitrarily close to $1/2$ using only verifiable messages, adding information to the announcement of the (true) ranking.

We first present a natural family of signaling strategies: a strategy is a partition of $[0,1]$ into n intervals defined by $n+1$ points $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_n = 1$. In addition to the announcement of the ranking, the seller announces (α_{i-1}, α_i) when $\alpha_{i-1} \leq v_\ell / v_h < \alpha_i$, where v_ℓ / v_h is the ratio of the low value to the high value. For example, for $n = 2$ and $\alpha_1 = 1/2$, this strategy achieves revenue equal to $5/12$. However, we show that with this family of signaling strategies the seller's revenue cannot exceed $4/9$ (which is $2/3$ of the social surplus). To achieve higher revenue we present a family of signaling strategies $\sigma(m,n)$, indexed by two integer parameters: A strategy is now given by mn intervals in $[0,1]$ defining m messages consisting of n intervals each, and the messages are nested in a specific way. We prove that with this family the bound of $1/2$ can be reached asymptotically: The limit of the seller's revenue as both m and n go to infinity is $1/2$. We conjecture that with partial truthful commitment power, the bound of $1/2$ can be achieved only asymptotically.

This phenomenon that the revenue achieved with full commitment power can be asymptoti-

[2]Depending upon what is revealed, additional information may not be revenue enhancing. See Andreoni, Che, and Kim (2007) and Kim and Che (2004).

cally achieved with partial truthful commitment power is not general; we show a counter example where this is not the case.

Finally, the study of the special case of two buyers with uniformly distributed values is mainly a matter of convenience and clarity of the presentation since the analysis is rather complex technically. Most of our results hold for any distributions of the values (whether symmetric or not) and more than two buyers. In Section 5, we outline in more detail how the approach and the results can be extended to more general auctions.

Related literature

Bergemann, Brooks, and Morris (2017) (BBM) also consider the possibility that, in a first-price auction, the buyers may have information about other buyers' valuations.[3] They consider a general information structure which is given as part of the description of the game.[4] In our approach, the information structure is a strategy of the seller, who is a player in the game.

As a player, the seller's objective is to choose the information structure (strategy) that results in the *maximum seller's revenue* (under constraints). When the seller has full commitment power to commit to a certain strategy, any strategy leads to a BBM model with an information structure induced by this strategy.

Thus, maximal revenue in the BBM model is achievable in our model when the seller has full commitment power by committing to the corresponding information structure. However, we show that maximal revenue may also be achieved by a seller with a less restrictive commitment power, namely, commitment to verifiability (a seller can commit to always send a message which is true but not to which truthful message to send). In particular, for the case of two buyers with independent uniformly distributed values (in $[0,1]$) while it follows from BBM that the maximum revenue is 1/2, we prove that in our model, this revenue is symptomatically achieved by a seller with verifiability commitment.

The now ubiquitous Bayesian Persuasion literature (Kamenica and Gentzkow 2011) has a sender (equivalent to a seller in our model) and a receiver (equivalent to a buyer).[5] The sender uses messages to influence the beliefs of the receiver and by doing so to influence the action of the receiver (whose preferences are not perfectly aligned with those of the sender). Applied to auctions, this has the seller sending information about the other buyers' valuations in order to increase bidding behavior (Sorokin and Winter 2021). See the proof of Claim 2 of this paper, for a simple example of how this may occur in auctions. We note that Bayesian persuasion requires full commitment power on behalf of the sender.

[3] See also Āzacis and Vida (2015), Fang and Morris (2006), and Bergemann, Brooks, and Morris (2013).

[4] In Figure 1 of BBM, all possible outcomes of a first-price auction (with two bidders and an independent, uniform distribution of values) resulting from different information structures are given both when buyers know their own values and when they do not.

[5] See more recently, Mathevet, Perego, and Taneva (2020). Also see Bergemann and Morris (2019) for a more general overview of the information design literature.

2 The model

The general idea of a seller having information about the buyers' private values and being able to communicate it, can be described by the following model.

The set of buyers is $N = \{1,\ldots,n\}$. Buyers have independent values $V_i \sim F_i$ with support $\mathbb{V}_i$ or more generally a joint distribution F on $\mathbb{V} = \times_{i=1}^{n}\mathbb{V}_i$. The information of the seller is a partition π of $\mathbb{V}$. The set of messages that is available to the seller to send to buyer i, which may depend on the real state of nature v is denoted by $A_i(v)$, and $A(v) = A_1(v) \times \ldots \times A_n(v)$ is the set of vectors of private signals to the buyers. A pure signaling strategy of the seller is $s : \mathbb{V} \to A(v)$, which is π-measurable, that is, $s(v)$ is constant on each partition element of π. This strategy means that buyer i receives (in addition to his value v_i) the signal $a_i \in A_i(v)$. Denote by S the set of pure signaling strategies. The set of *mixed* signaling strategies is $\Delta(S)$ and the set of *behavioral* strategies is $\Sigma = \{\sigma : \mathbb{V} \to \Delta(A(v)),\ \ \pi - \text{measurable}\}$. Since in our game the seller has one move, it is trivially a game of perfect recall and hence we can restrict attention to behavioral strategies. Furthermore, for our main results (existence and asymptotic optimality) the seller uses only pure strategies (which strengthens our results).

In the present work, we confine the seller to send only *public messages*; $s(v) = (a,\ldots,a)$ where $a \in \cap_{i\in N} A_i(v)$, as our main results can be obtained under this restriction.[6] Therefore, it will be convenient to modify our notation slightly: We write $s(v) = a$ and, $A(v)$ will denote the set of available public messages at v.

Assuming from now on independent values, the game is played as follows. Nature chooses $v = (v_1,\ldots,v_n) \in \mathbb{V}$ according to $F_1 \times \cdots \times F_n$. The seller is informed of $\pi(v)$ (the partition element containing v). Buyer i receives the message v_i (his private value). The seller sends a (public, pure) message $s(v) = a \in A(v)$. Each buyer i bids $b_i(v_i,a)$ in a sealed-bid first-price auction.

Without loss of generality we may assume that a message in $A(v)$ is a subset of $\mathbb{V}$; by replacing any message $\tilde{a} \in A(v)$ with $a := \{v \in \mathbb{V} | \tilde{a} \in A(v)\} \subset \mathbb{V}$. The commitment power levels can now be formally defined as follows:

NC: The game with a seller that has *no commitment power* is a game in which $A(v) = 2^{\mathbb{V}}$ for all $v \in \mathbb{V}$, that is, at any v, the seller can announce any subset of $\mathbb{V}$.

VC: The game with a seller endowed with *partial truthful commitment power* is the game in which for all $v \in \mathbb{V}$, $A(v) = \{V \subset \mathbb{V} | \pi(v) \subset V\}$, that is, the seller can send any information that is compatible with his information $\pi(v)$. In other words, the subset of $\mathbb{V}$ announced by the seller must contain his true information $\pi(v)$ but may contain additional values (potentially making it less informative). Equivalently, the seller is restricted to *verifiable* messages thereby cannot contain an incorrect statement. (Hence, the abbreviation VC - commitment to verifiable messages.)

FC: The game with *full commitment power* is a different auction mechanism played as follows:

- The seller submits (publicly) a π-measurable signaling strategy $s : \mathbb{V} \to 2^{\mathbb{V}}$.

[6] In a later example, the seller uses a mixed public message; in that case $a \in \cap_{i\in N} \Delta(A_i(v))$.

– Nature chooses $v \in \mathbb{V}$ according to $F_1 \times \cdots \times F_n$.

– Each buyer i is informed of v_i and $s(v)$ is announced publicly (by a neutral auctioneer).

– Each buyer i submits a sealed bid $b_i(v_i, s(v))$. The buyer with the highest bid wins the object and pays his bid (with the provision of some tie breaking rule).

Having set the general model, in this paper we consider the case of two buyers which yields a three player game. Although this three-player game with incomplete information is played sequentially and the two buyers form their beliefs from their private values and the (public) message of the seller, we refer to its equilibrium simply as *Bayes-Nash equilibrium* although it can also be viewed as a *Sequential Equilibrium* of the extensive form game.

3 Verifiable continuous messages

3.1 A bound on the revenue

We begin by establishing a bound on the maximum obtainable revenue. Given the buyers' values v_1 and v_2, we denote the lower and higher values as v_ℓ and v_h, respectively.[7]

Theorem 1. *When the values are iid and $v_i \sim U[0,1]$, the revenue of the seller cannot exceed $\frac{3}{4}E(v_h) = \frac{1}{2}$ (with any level of commitment power, set of signals, and any level of information).*

We prove this theorem by looking at the *Maxmin* of each buyer in this game, that is, the minimum profit a buyer can guarantee. This is given in the following lemma.

Lemma 1. *Each buyer can guarantee a profit of at least $\frac{1}{8}E(v_h) = \frac{1}{12}$.*

Proof. Given that no buyer is bidding more than his value, the worst case scenario for buyer i is when the other buyer bids his value.[8] By bidding $\frac{1}{2}v_i$, buyer i would therefore win with probability of at least $\frac{1}{2}$ when he is the high-value buyer. Thus, contingent on having the highest value v_h, buyer i can guarantee $\frac{1}{2}(v_h - \frac{1}{2}v_h) = \frac{1}{4}v_h$. Since buyer i has a 1/2 chance of having the high value, he can guarantee an ex-ante profit of at least $\frac{1}{8}E(v_h) = \frac{1}{12}$. □

Proof of Theorem 1.

Since the equilibrium payoff to any player is at least his Maxmin payoff, it follows from Lemma 1 that, in equilibrium, the sum of the buyers' profits is at least $\frac{1}{4}E(v_h)$. Therefore, the seller's revenue cannot exceed $\frac{3}{4}E(v_h) = \frac{1}{2}$. □

Remark 1. *Note that as the seller becomes a strategic player, the Maxmin concept is naturally called for since, when the auction is efficient, this is a constant-sum game between the seller and the high-value buyer.*

Remark 2. *We note that the Maxmin strategy of the buyers namely, 'bid half of your value'*

[7] A similar result is also obtained in Bergemann, Brooks, and Morris (2017).

[8] In a standard first-price auction, without this assumption of not bidding above one's value, we may get multiple equilibria. See Kaplan and Zamir (2015).

is exactly the equilibrium strategy in the standard model without information. In other words, the Maxmin strategy of each buyer, is just to ignore the message of the seller and follow the equilibrium strategy in the standard model.

This is not always the case. It is straightforward to show that in the standard two-buyer first-price auction with values drawn independently from cdf $F(v)$, the equilibrium bid function is $b(v) = v - \frac{\int_0^v F(x)dx}{F(v)}$ while the Maxmin strategy is the inverse function of $v(b) = \frac{F(b)}{F'(b)} + b$. When $F(v) = 2v - v^2$, the equilibrium bid is $b(v) = \frac{v(3-2v)}{3(2-v)}$ and the Maxmin strategy is the inverse of $v(b) = \frac{4b-3b^2}{2-2b}$. These two functions are not inverses of each other: $b(\frac{1}{2}) = \frac{2}{9}$ while $v(\frac{2}{9}) = \frac{10}{21}$.

3.2 Full commitment may do strictly better than verifiability commitment

While the focus of our paper is revenue achievable with verifiability commitment, we first establish that there is a significant distinction between FC and VC.[9]

Consider two buyers, A and B, where buyer A has a value of 1, while buyer B can have a value of either $\frac{1}{4}$ or $\frac{3}{4}$ with equal probability. The seller knows buyer B's value and can say either $\frac{1}{4}$, or $\frac{3}{4}$ or ϕ (nothing). Formally, ϕ is the uninformative message of $\{\frac{1}{4}, \frac{3}{4}\}$ meaning: "B's value is either $\frac{1}{4}$ or $\frac{3}{4}$."

Claim 1. *Under verifiability commitment, the seller will fully reveal the value of buyer B and thus earn expected revenue equal to* $1/2$.

Proof. A general behavioral strategy of a seller with verifiability commitment is:

1. When B's value is $\frac{1}{4}$ send $\frac{1}{4}$ or ϕ with probabilities α and $1-\alpha$, respectively.
2. When B's value is $\frac{3}{4}$ send $\frac{3}{4}$ or ϕ with probabilities β and $1-\beta$, respectively.

If $\beta < 1$, then in equilibrium, A must be bidding $3/4$ when ϕ is sent. Buyer A cannot be bidding strictly higher and if A is bidding strictly is lower, then for the seller, sending ϕ would be dominated by sending $3/4$ yielding revenue equal to $3/4$. Consequently, $\alpha = 0$, since the seller can obtain $3/4$ when the value is $1/4$ by sending ϕ while sending $1/4$ yields revenue equal to $1/4$. But, if $\alpha = 0$, then when ϕ is sent, $P(\frac{1}{4}|\phi) = \frac{\frac{1}{2}}{\frac{1}{2}+\frac{1-\beta}{2}} = \frac{1}{2-\beta}$. Therefore, when ϕ is sent, buyer A has incentive to deviate and bid $1/4$ (rather than $3/4$) since

$$\frac{3}{4} \cdot \frac{1}{2-\beta} > \frac{1}{4},$$

a contradiction establishing that $\beta < 1$ is not possible in equilibrium.

If $\beta = 1$, then when ϕ or $1/4$ is sent, buyer A can deduce that the value is $1/4$. Hence, any equilibrium involves full revelation and consequently a revenue equal to $1/2$. □

Claim 2. *The seller can achieve higher revenue under full commitment than under verifiability commitment.*

[9]See Hart, Kremer, and Perry (2017) for a class of games where the equilibria of FC and VC coincide.

Proof. With full commitment power, the seller can commit to use the following strategy: When buyer B's value is $3/4$, the seller sends the message ϕ. When buyer B's value is $1/4$, the seller sends the message ϕ with probability $1/2$ and message $1/4$ with probability $1/2$. The equilibrium of the game induced by this signaling strategy is that in which buyer A bids $1/4$ when hearing $1/4$ (probability of $1/4$), winning with probability 1 and generating a profit of $3/4$. When hearing ϕ (probability of $3/4$), buyer A bids $3/4$ and wins with probability 1 generating a profit of $1/4$. Buyer A has no profitable deviation since when hearing ϕ, the conditional probability that B's value is $1/4$ is $1/3$. Thus bidding $1/4$ when hearing ϕ yields buyer A expected profit of $1/3 \times 3/4 = 1/4$, which is equal to the profit when bidding $3/4$. The revenue of the seller in this equilibrium is

$$\frac{3}{4} \times \frac{3}{4} + \frac{1}{4} \times \frac{1}{4} = \frac{5}{8} > \frac{1}{2}.$$

□

Remark 3. *Note that given the strategy of buyer A to bid $3/4$ when hearing ϕ, the seller could profitably 'deviate' from his strategy by sending ϕ always when B's value is $1/4$. However such a 'deviation' is not possible in full commitment, while it is allowed in verifiability commitment (preventing such a strategy in equilibrium).*

Claim 3. *In this example, with full commitment, the seller achieves the highest possible revenue.*

Proof. As we saw, under full commitment the seller gets a revenue equal to $\frac{3}{4} \cdot \frac{3}{4} + \frac{1}{4} \cdot \frac{1}{4} = \frac{5}{8}$. To see that this is the highest revenue possible, we invoke again the Maxmin argument: Buyer A can bid $1/4$ and guarantee himself a payoff of $3/4$ with probability $1/2$ which is $3/8$ in expectation. Since the total maximum surplus is 1, the seller cannot extract a revenue higher than $5/8$. □

3.3 Ranking or Values

We provide now an example where the seller increases revenue with verifiable messages and simple signalling strategies. In an auction with two buyers with values distributed uniformly and independently in $[0,1]$, the seller knows the realized values (v_1, v_2) and can send one of the following three (public) signals: (1) The uninformative signal ϕ. (2) Announce the true ranking $r \in \{r_1, r_2\}$ (where $r_1 = \{v \in \mathbb{V} | v_1 \geq v_2\}$ and $r_2 = \{v \in \mathbb{V} | v_2 > v_1\}$). (3) Announce the true values (v_1, v_2). Denote v_ℓ and v_h as the low and high values, respectively.

Theorem 2. *When values are iid and $v_i \sim U[0,1]$, there exists an equilibrium where (see Figure 2):*

(i) The seller announces r if $v_\ell \leq \frac{1}{2} v_h$ and (v_1, v_2) if $v_\ell > \frac{1}{2} v_h$.

(ii) The low-value buyer bids his value, $b_\ell(v_\ell) = v_\ell$ if either r or (v_1, v_2) is announced (note that the ranking is common knowledge in both cases).[10]

(iii) The high-value buyer bids $\frac{1}{2} v_h$ if r is announced and v_ℓ if (v_1, v_2) is announced.

[10] For convenience, assume that the buyer with the highest value wins in the case of a tie.

(iv) If ϕ is announced, each buyer bids $1/2$ his value.

Proof. Straightforward verification shows that no buyer or seller has a profitable deviation. $\square$

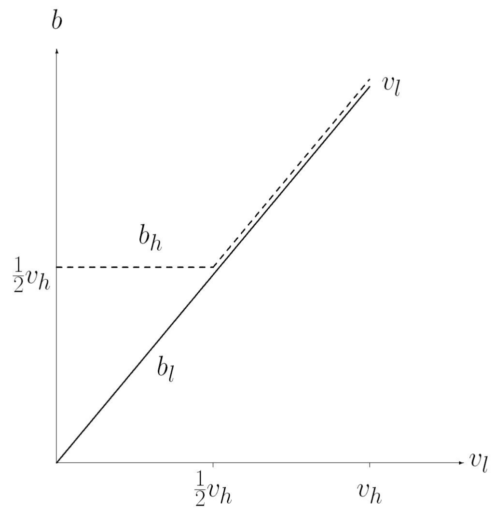

FIGURE 2. *Revenue equals $\frac{5}{12} = 0.4167$ (which is $\frac{5}{8}$ of the social surplus $\frac{2}{3}$)*

Remark 4. *This equilibrium obtains revenue equal to $\frac{5}{12}$, which is $\frac{5}{8}$ of the social surplus $\frac{2}{3}$. This compares with the standard equilibrium that generates revenue equal to $\frac{1}{3}$, which is half of the social surplus of $\frac{2}{3}$.*

Remark 5. *The resulting allocation is efficient.*

Remark 6. *If the seller always uses option (2) - amounting to the ranking, the induced game between the buyers is that studied in Landsberger et al. (2001) yielding a revenue equal to 0.3969 to the seller. However, this is not an equilibrium with VC. Note that announcing r yields a revenue equal to $\max\{b_\ell(v_\ell), b_h(v_h)\}$ (see Figure 1). When v_ℓ is sufficiently close to v_h, we have $v_\ell > \max\{b_\ell(v_\ell), b_h(v_h)\}$. Therefore, announcing (v_1, v_2) would be a profitable deviation for the seller as it would yield a revenue equal to v_ℓ.*

Remark 7. *Commitment to always announcing r is not part of an equilibrium with FC since committing to the strategy in Theorem 2 yields revenue equal to $5/12$, which is higher than revenue generated by always sending r and generating revenue equal to 0.3969.*

3.4 Continuous range messages

We now extend the set of messages available for the seller from those assumed in the previous subsection to a larger set that we call continuous range messages.

For $n \geq 1$, let $0 = d^0 < d^1, \ldots, < d^n = 1$ which defines the following partition of $[0,1]$: $\pi(n) = \{[0,d^1], (d^1,d^2], \ldots, (d^i,d^{i+1}], \ldots, (d^{n-1},1]\}$. Given his information (v_1,v_2), the seller has the following options for messages:

Option (1) the empty message ϕ (no announcement),

Option (2) the true values (v_1, v_2), or

Option (3) the ranking r and the partition element (the interval) containing v_ℓ / v_h.

When $n = 1$, we have $\pi(1) = \{[0,1]\}$ and option (3) is equivalent to just sending the ranking r. The equilibrium described in Theorem 2 corresponds to the case $n = 2$ and $\pi(2) = \{[0, \frac{1}{2}], (\frac{1}{2}, 1]\}$, where the seller uses option (3) when $v_\ell \leq \frac{1}{2} v_h$ and option (2) when $v_\ell > \frac{1}{2} v_h$.

We will now show that for any n there is an equilibrium in which the seller uses option (2) whenever $v_\ell > d^{n-1} v_h$ and option (3) otherwise. We then will proceed to find an equilibrium of this form (namely, determining the values of d^i) that maximizes revenue.

Lemma 2. *In any equilibrium under verifiability commitment with these three options in which the seller uses option (3) for all $v_\ell \in (d^{i-1} v_h, d^i v_h]$, the revenue (given that $v_\ell \in (d^{i-1} v_h, d^i v_h]$) and hence the high buyer's bid b_h must be equal to $d^i v_h$.*

Proof. If the expected revenue x where $x < d^i v_h$, then with positive probability we have $x < v_\ell \leq d^i v_h$ and in this event it is a profitable deviation for the seller to announce (v_1, v_2) (with revenue v_ℓ) rather $(d^{i-1}, d^i]$ (with revenue x). □

The following corollary is a consequence of Lemma 2.

Corollary 1. *When only options (2) and (3) are used by the seller, the high buyer wins the auction and hence the auction is efficient.*

The next lemma sets further conditions on the structure of these messages to support such an equilibrium.

Lemma 3. *When values are iid with the uniform distribution on $[0,1]$, if a message $m = (d^{i-1}, d^i]$ is announced for all $v_\ell \in (d^{i-1} v_h, d^i v_h]$, then $d^i \leq (1 + d^{i-1})/2$.*

Proof. The high-value buyer's maximization problem is to chose b so as to maximize the function $f(b)$ given by

$$f(b) := P(v_\ell < b | v_\ell \in I) \cdot (v_h - b) = \frac{|[0,b] \cap I|}{|I|} \cdot (v_h - b) = \frac{b - d^{i-1} v_h}{v_h (d^i - d^{i-1})} \cdot (v_h - b).$$

Since by Lemma 2, the high-value buyer bids $d^i v_h$ which is the maximum of the interval I, it must be that $f'(b) \geq 0$ at $b = d^i v_h$ (otherwise the profit could be increased by slightly lowering the bid). Since $f'(b)$ is non-negative only when $b \leq v_h (1 + d^{i-1})/2$, it follows that $d^i \leq (1 + d^{i-1})/2$. □

Theorem 3. *For any n, the highest revenue is attained when $d^i = 1 - (\frac{1}{2})^i$ for $i = 1, \ldots, n-1$. In the limit, as $n \to \infty$, the highest revenue equals $\frac{4}{9}$. (See Figure 3.)*

Proof. Denote by $d^*(n) := (d^{i*})_{i=0}^n$ the points defining a partition which maximizes revenue.

By Lemma 3 we have:

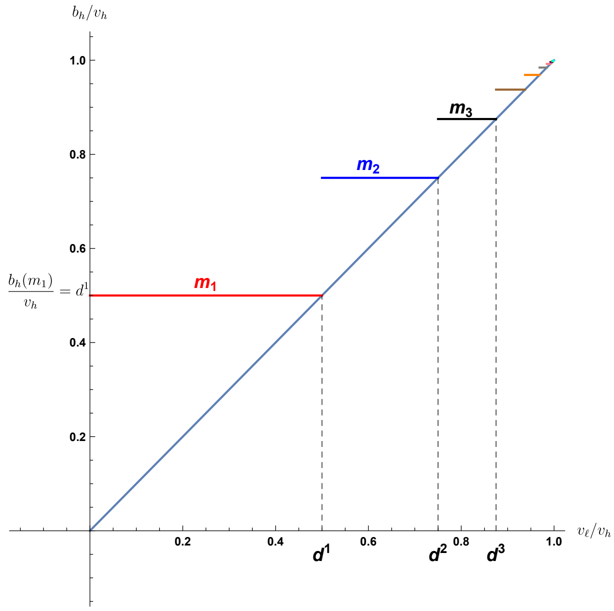

FIGURE 3. *Seller's messages m_i. The bid of the H-buyer when message m_i is announced equals v_h times the right-hand point d^i of message m_i. The auction is efficient and the revenue equals $\frac{4}{9}$ (which is $\frac{2}{3}$ of the "social surplus" $\frac{2}{3}$).*

$$d^{i*} \leq \frac{1+d^{i-1*}}{2}; \quad i = 1,\dots,n-1. \tag{1}$$

Therefore, the proof is completed if we show that Equations (1) are satisfied as equalities and thus determining d^{i*} inductively from $d^{0*} = 0$.

Step 1: Any $d^*(n)$ that maximizes revenue must satisfy $d^{1*} = 1/2$.

Proof. As a function of the partition $d^*(n)$ and v_h, the revenue $R(d^*(n), v_h)$ is given by,

$$R(d^*(n), v_h) = [\sum_{1}^{n-1} d^{i*}(d^{i*} - d^{i-1*}) + \frac{1-(d^{n-1*})^2}{2}]v_h. \tag{2}$$

The first term in the RHS is, by Lemma 2, the revenue when $0 \leq v_\ell \leq d^{n-1*}v_h$. The second term is the revenue when $d^{n-1*}v_h < v_\ell \leq 1$. In this event, whose probability is $(1-d^{n-1*})$, the seller's message is (v_ℓ, v_h) and the revenue is v_ℓ. Thus, the (conditional) average revenue in this interval is $(1+d^{n-1*})/2$, and hence the last term in (2).

For fixed $(d^{i*})_{i=2}^{n}$, the revenue $R(\hat{d}^1)$ as a function of the first point of the partition $\hat{d}^1$ is

$$[\sum_{i=3}^{n-1} d^{i*}(d^{i*} - d^{i-1*}) + (\hat{d}^1)^2 + d^{2*}(d^{2*} - \hat{d}^1) + \frac{1-(d^{n-1*})^2}{2}]v_h. \tag{3}$$

To maximize revenue, $\hat{d}^1$ has to maximize $f(\hat{d}^1) := (\hat{d}^1)^2 + d^{2*}(d^{2*} - \hat{d}^1)$ of the RHS, in the range,

$$0 < \hat{d}^1 < d^{2*}, \tag{4}$$

and under the constraints (1),

$$2d^{2*} - 1 \le \hat{d}^1 \le \frac{1}{2}. \tag{5}$$

Note there are strict inequalities in (4), since the intervals are non degenerate.

In maximizing $f(\hat{d}^1)$, we first establish that $1/2 < d^{2*} \le 3/4$. The right hand inequality is obtained by applying the bound (1) for $i = 1$ to obtain $\hat{d}^1 \le 1/2$ and then for $i = 2$ to obtain $d^{2*} \le 3/4$. To prove the left-hand-side inequality we note that if $d^{2*} \le 1/2$, then the bounds (5) are implied by (4), and consequently the maximum of $f(\hat{d}^1)$ is attained at $\hat{d}^1 = 0$ or at $\hat{d}^1 = d^{2*}$, contradicting (4). Now, as $d^{2*} > 1/2$, the bounds (5) are both in the range (4) and therefore the maximum of the parabola $f(\hat{d}^1)$ is attained either at $\hat{d}^1 = 2d^{2*} - 1$ or $\hat{d}^1 = 1/2$. Finally, we show that the maximum is attained at $\hat{d}^1 = 1/2$, since

$$f(1/2) - f(2d^{2*} - 1) = -2(d^{2*})^2 + \frac{5}{2}d^{2*} - \frac{3}{4},$$

is strictly positive for $1/2 < d^{2*} < 3/4$ and 0 for $d^{2*} = 3/4$, in which case $2d^{2*} - 1 = \frac{1}{2}$. Hence, $\hat{d}^1 = 1/2$ is the only maximum of $f(\hat{d}^1)$, completing the claim in Step 1.

Step 2: The Equations (1) are satisfied with equalities for all i.

Proof. In Step 1, we showed that Equations (1) is satisfied with equality for $i = 1$. Assume by contradiction that it is not satisfied with equality for a particular j, where $1 < j \le n - 1$, that is, $d^{j*} < (1 + d^{j-1*})/2$. Since all intervals are non degenerate, we have (using Step 1 again), $d^{j*} > d^{1*} = 1/2$. We claim that increasing d^{j*} so as to satisfy equality in (1) will increase revenue. Let us start for $j < n - 1$. Define a new sequence $\hat{d}(n)$ by $\hat{d}^i = d^{i*}$ for all $i \ne j$ and $\hat{d}^j$ to be determined later. Then, by (2), the revenue of this new sequence as a function of $\hat{d}^j$ is

$$R(\hat{d}(n), v_h) = [\sum_{\substack{i=1 \\ i \ne j \\ i \ne j+1}}^{n-1} d^{i*}(d^{i*} - d^{i-1*}) + \hat{d}^j(\hat{d}^j - d^{j-1*}) + d^{i+1*}(d^{j+1*} - \hat{d}^j) + \frac{1 - (d^{n-1*})^2}{2}]v_h. \tag{6}$$

Thus, maximizing revenue amounts to maximizing $f(\hat{d}^j) := \hat{d}^j(\hat{d}^j - d^{j-1*}) + d^{i+1^*}(d^{j+1*} - \hat{d}^j)$ as a function of $\hat{d}^j$. The derivative of f is $f'(\hat{d}^j) = 2\hat{d}^j - d^{j+1*}$. Since $d^{j*} > 1/2$ and $d^{j+1*} \le 1$, the derivative $f'(\hat{d}^j)$ is strictly positive at d^{j*} and hence the revenue strictly increases by increasing d^{j*}, reaching its maximum (subject to the constraints (1)) at $\hat{d}^j = \frac{1+d^{j-1*}}{2}$. For $j = n - 1$ revenue is

$$R(\hat{d}(n), v_h) = [\sum_{i=1}^{n-1} d^{i*}(d^{i*} - d^{i-1*}) + \hat{d}^{n-1}(\hat{d}^{n-1} - d^{n-2*}) + \frac{1 - (\hat{d}^{n-1})^2}{2}]v_h. \tag{7}$$

This function is strictly increasing in $\hat{d}^{n-1}$ since the derivative is $\hat{d}^{n-1} - d^{n-2*}$ which is strictly positive. Thus, the constraint in (1) for $i = n-1$ is binding - concluding the proof of Step 2.

We have now shown that constraints (1) hold with equality and thus $d^{i*} = 1 - (\frac{1}{2})^i$ for $i = 1, \ldots, n-1$. Substituting this expression for d^{i*} into (2), yields revenue as:

$$R(d^*(n), v_h) = [\sum_{i=1}^{n-1} (\frac{1}{2})^i (1 - (\frac{1}{2})^i) + \frac{1 - (1 - (\frac{1}{2})^{n-1})^2}{2}] v_h = [\frac{2}{3} - \frac{1}{6}(\frac{1}{4})^{n-1}] v_h, \tag{8}$$

from which it follows that $\lim_{n\to\infty} R(d^*(n), v_h) = \frac{2}{3} v_h$. Finally, integrating over v_h, (recalling that the density of v_h is $2v_h$), we get:

$$\lim_{n\to\infty} R(d^*(n)) = \int_0^1 \lim_{n\to\infty} R(d^*(n), v_h) \cdot 2v_h dv_h = \int_0^1 \frac{2}{3} v_h \cdot 2v_h dv_h = \frac{4}{9},$$

concluding the proof of Theorem 3. □

Remark 8. *The revenue equal to 4/9 obtained with continuous messages is strictly higher than revenue equal to 5/12 obtained with only using messages of the ranking or values but less than the theoretical maximum revenue equal to 1/2.*

3.5 Non-optimality of continuous messages

We now show that simple messages consisting of single intervals considered previously do not obtain the highest revenue by way of an example.

Theorem 4. *Higher revenue can be generated by signals of the form $v_\ell \in S$ where S is a discontinuous set (union of intervals).*

Proof:

Assume that in addition to sending the ranking and values the seller chooses $(\alpha_1, \alpha_2, \alpha_3)$ such that $0 < \alpha_1 < \alpha_2 < \alpha_3 < 1$ and applies the following strategy: send (m_1), if $\alpha_1 v_h < v_\ell \le \alpha_2 v_h$; send (m_2), if $v_\ell \le \alpha_1 v_h$ or $\alpha_2 v_h < v_\ell \le \alpha_3 v_h$; send (v_ℓ, v_h), if $\alpha_3 v_h < v_\ell \le v_h$.

The parameters $0 < \alpha_1 < \alpha_2 < \alpha_3 < 1$ are chosen so that in equilibrium: the high-value buyer bids $\alpha_2 v_h$ following (m_1); the high-value buyer bids $\alpha_3 v_h$ following (m_2); the high-value buyer bids v_ℓ following message (v_ℓ, v_h).

This implies the following three inequalities to be satisfied by $\alpha_1, \alpha_2, \alpha_3$:

(A) $\alpha_3 - (\alpha_2 - \alpha_1) \le 1 - \alpha_3$,

(B) $\alpha_2 - \alpha_1 \le 1 - \alpha_2$,

(C) $1 - \alpha_3 \ge (1 - \alpha_1) \cdot \frac{\alpha_1}{\alpha_3 - (\alpha_2 - \alpha_1)}$.

Inequality (A) guarantees that the high value buyer does not lower his bid (slightly) from $\alpha_3 v_h$ when m_2 is sent. Inequality (B) guarantees that the high value buyer does not lower his bid (slightly) from $\alpha_2 v_h$ when m_1 is sent. Inequality (C) guarantees that the high value buyer does not lower his bid from $\alpha_3 v_h$ to $\alpha_1 v_h$ when m_2 is sent.

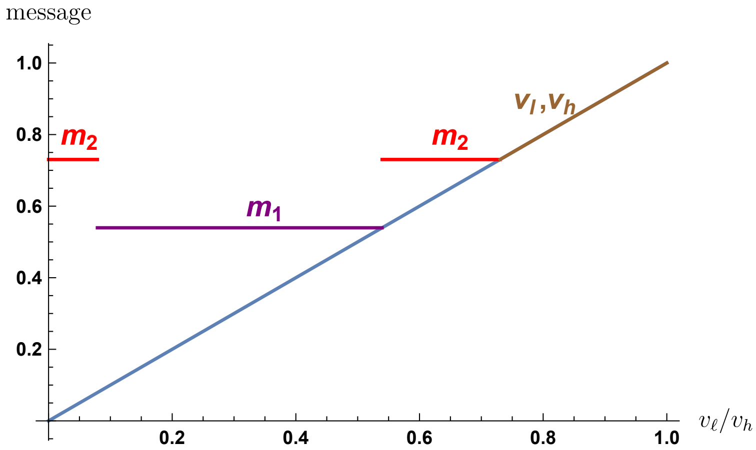

FIGURE 4. *Messages of discontinuous range generating revenue* $\approx 0.6909E(v_h) > \frac{2}{3}E(v_h) = 4/9$.

Setting inequalities (A)-(C) to equalities and solving yields $(\alpha_1, \alpha_2, \alpha_3) \approx (0.079, 0.5395, 0.7303)$ and the resulting messaging strategy is given in Figure 4 yielding revenue strictly higher than $\frac{4}{9}$ from Theorem (3).

□

Theorem 4 indicates there is a need to study messages beyond simple continuous ones. We do so in the next section.

4 Higher revenue by verifiable non-continuous messages

In this section, we show that under verifiability commitment it is possible to obtain, asymptotically, the maximal revenue equal to $\frac{3}{4}E(v_h) = \frac{1}{2}$, using a special set of signalling strategies.

4.1 The set of signalling strategies

We consider the following two-parameter family of signaling strategies of the seller denoted by $\sigma(m,n)$ where m and n are positive integers. The strategies have the following form:

- There are m messages where each message is a subset of $[0,1]$ consisting of n intervals (segments).
- $x = (x_1^1, x_1^2 \ldots, x_1^m, \ldots, x_n^1, x_n^2 \ldots, x_n^m)$ is a partition of the interval $[0, d^m]$ into $m \cdot n$ intervals (segments), where $0 < d^m < 1$ will be specified later. For convenience we shall assume that all segments are of the form $x_j^i = (\cdot, \cdot]$ (that is, open at the left end and closed at the right end) except x_1^1 which is closed at both ends since it starts at 0, that is, $x_1^1 = [0, \cdot]$.
- For $1 \leq i \leq m$, let $x^i = \bigcup_{j=1}^n x_j^i$ be the union of the n segments composing message i, which we shortly call 'message i'.
- Thus, x_j^i is the j-th interval of message i and our notation above; $x = (x_1^1, x_1^2 \ldots, x_1^m, \ldots, x_n^1, x_n^2 \ldots, x_n^m)$

indicates the way the messages are interlaced: Starting from 0, the intervals begin with the first intervals of all messages in the order $1,2,\ldots,m$, then are the second intervals of all messages in the same order, etc.

- Equivalently, the intervals appear from left to right in the lexicographic order starting with the 2nd coordinate, that is, x_r^k appears before x_j^i *iff* $r < j$ or $r = j$ and $k \leq i$. We represent this order as $(k,r) \preceq (i,j)$.

The signaling strategy of the seller is:

- When the values (v_1, v_2) are realized, the seller announces (publicly) the ranking.
- Denoting the ranked values by (v_ℓ, v_h), if $v_\ell/v_h \in x^i$, the seller announces also (publicly) the message i.
- We denote by $p(i) = Pr\{v_\ell/v_h \in x^i\}$ the probability that message i is announced, which is $p(i) = |x^i|$, the total measure of message i.
- If $v_\ell/v_h > d^m$ the seller announces (publicly) the two values (v_1, v_2).

4.2 Induced game equilibrium

Any such strategy induces a game between the two buyers and we will look at the Bayes-Nash equilibria of that game and the corresponding seller's revenue. We point out that unless the signaling strategy maximizes the seller's revenue over all (verifiable) signaling strategies (not necessarily in the $\sigma(m,n)$ family), an equilibrium of the induced game *is not* an equilibrium of the three players game including the seller as a player. Nevertheless, for convenience, we use the term 'equilibrium' also for the induced games.

Given a strategy $\sigma(m,n)$ we denote by d^i the right-end point of the message i (that is $d^i = \max\{x | x \in x^i\}$). The partition x of the strategy $\sigma(m,n)$ will be chosen so the following bidding strategies of the buyers will be a Bayes-Nash equilibrium in the induced game which will be denoted by $E(m,n)$.

(i) When the ranking and message i is announced by the seller, the high-value buyer bids $b_h(i) = d^i \cdot v_h$ and the low-value buyer bids his value.
(ii) When the values (v_1, v_2) are announced, both buyers bids v_ℓ and the high-value buyer wins.[11]

For our computations and proofs, we need additional notation:

- $(c_j^i, d_j^i] := x_j^i$, that is, c_j^i, d_j^i are the bounds of interval x_j^i.
- $\ell_j^i := |x_j^i| = d_j^i - c_j^i$ is the measure of interval x_j^i.
- $d^i = d_n^i$ is the endpoint of the i-th message, that is, the endpoint of its last interval.

[11] This statement is used here for simplicity. To make it precise, avoiding tie-breaking issues, we can have the low value buyer bid uniformly in $[v_\ell - \varepsilon, v_\ell]$ for some small $\varepsilon < v_h - v_\ell$. This will be still an equilibrium except for the probability zero event $v_h = v_\ell$.

- The probability of message i is therefore also given by

$$p(i) = |x^i| = \sum_{j=1}^{n} \ell^i_j. \tag{9}$$

Remark 9. *As we mentioned, the right-end point d^i_j belongs to the message i while left-end point c^i_j belongs to the message whose interval ends at c^i_j (except $i = j = 1$ where $x^1_1 = [0, d^1_1]$), and by the structure of $\sigma(m,n)$ the endpoints d^i_j of the segments are given by*

$$d^i_j = \sum_{(k,r)\preceq(i,j)} \ell^k_r \quad \text{for } i = 1,\dots,m;\ j = 1,\dots,n. \tag{10}$$

In particular, the endpoints d^i of the messages (which will play a special role) are:

$$d^i = d^i_n = \sum_{(k,r)\preceq(i,n)} \ell^k_r \quad \text{for } i = 1,\dots,m. \tag{11}$$

Remark 10. *Note that by this special structure, the vector $D := \{d^i_j | i = 1,\dots,m;\ j = 1,\dots,n\}$ of right endpoints of the segments determines the signaling strategy $\sigma(m,n)$ completely. Indeed, the left endpoints c^i_j are obtained from D by*

$$\begin{aligned} c^1_1 &= 0 \\ c^1_j &= d^m_{j-1} \quad \text{for} \quad 2 \le j \le n \\ c^i_j &= d^{i-1}_j \quad \text{for} \quad 2 \le i \le m;\ 1 \le j \le n, \end{aligned} \tag{12}$$

and the length of the intervals ℓ^i_j are obtained from Equation (10) as follows

$$\begin{aligned} \ell^1_1 &= d^1_1 \\ \ell^1_j &= d^1_j - d^m_{j-1} \quad \text{for} \quad 2 \le j \le n \\ \ell^i_j &= d^i_j - d^{i-1}_j \quad \text{for} \quad 2 \le i \le m;\ 1 \le j \le n. \end{aligned} \tag{13}$$

Remark 11. *Note that this structure implies that the last segments of the messages are lined up from left to right:*

$$d^1 = d^1_n = c^2_n < d^2_n = d^2 = c^3_n < d^3_n = d^3 = \ldots < d^m_n = d^m$$

In particular we have the monotonicity of the right-end-points of the messages which will be used later:

$$0 < d^1 < d^2 < \ldots < d^m < 1. \tag{14}$$

Thus, a seller's signaling strategy $\sigma(m,n)$ is defined by the vector of $m \cdot n$ numbers in $[0,1]$, $D = \{d^1_j | i = 1,\dots,m;\ j = 1,\dots,n\}$, or equivalently by the vector $L := \{\ell^1_j | i = 1,\dots,m;\ j = 1,\dots,n\}$.

We now find the necessary conditions on the array D to guarantee that the pair of buyers' strategies $E(m,n)$ (described on page 15) will indeed be a Bayes-Nash equilibrium in the game induced by the signaling strategy $\sigma(m,n)$. For the clarity of the representation, these conditions will involve also the array L (related to D by Equations (13)).

To find the conditions to prevent a profitable deviation for the H-buyer (high-value buyer) when receiving message i and bidding $b_h(i) = d^i \cot v_h$, we proceed as follows:

1. Observe first that any bid not in the range of the message $x^i = \bigcup_{j=1}^{n}(c^i_j, d^i_j]$ is strictly dominated: Since a bid b such that $d^i_j < b < c^i_{j+1}$ is strictly dominated by the bid $b' = d^i_j \cdot v_h$ which has the same probability of winning as b and pays lower price when winning the object.

2. Denote by $u(b|v_h,i)$ the expected profit of the H-buyer with value v_h when receiving message i and bidding b. We impose now the equilibrium condition that, given a message i, the best reply of the H-buyer must be one of the right-end points of segments of that message. Namely, bidding in the open interval (c^i_j, d^i_j) is strictly dominated by the bidding $d^i_j \cdot v_h$. This condition means that for any message $1 \le i \le m$ and any segment j of that message, $1 \le j \le n$, the strict inequality

$$u(d^i_j \cdot v_h - \varepsilon | v_h, i) < u(d^i_j \cdot v_h | v_h, i) \text{ must hold for any } 0 < \varepsilon < \ell^i_j. \tag{15}$$

   Since

$$u(d^i_j \cdot v_h - \varepsilon | v_h, i) = \frac{(\sum_{r=1}^{j} \ell^i_r) - \varepsilon}{\sum_{r=1}^{n} \ell^i_r}(1 - d^i_j + \varepsilon)v_h \text{ and } u(d^i_j \cdot v_h | v_h, i) = \frac{\sum_{r=1}^{j} \ell^i_r}{\sum_{r=1}^{n} \ell^i_r}(1 - d^i_j)v_h,$$

   Equation (15) implies

$$\sum_{r=1}^{j} \ell^i_r < (1 - d^i_j) + \varepsilon \text{ for all } \varepsilon > 0, 1 \le i \le m,\ 1 \le j \le n, \tag{16}$$

   which is equivalent to:

$$\sum_{r=1}^{j} \ell^i_r \le (1 - d^i_j) \text{ for } 1 \le i \le m,\ 1 \le j \le n. \tag{17}$$

3. Since in the above Inequality (17), the left-hand side increases in j and the right-hand side decreases in j by (14), the inequality holding for $j = n$ is enough to guarantee that it holds for all j. Among the signalling strategies D satisfying the necessary condition, let the seller select the one that the inequality for $j = n$ holds with equality, that is:[12]

$$\sum_{r=1}^{n} \ell^i_r = (1 - d^i_n) = (1 - d^i),\ \ 1 \le i \le m. \tag{18}$$

   Note that Equation (18) means that when receiving a message i, the (conditional) expected payoff of the H-buyer when bidding $d^i \cdot v_h$, (and hence winning the object with probability 1), is equal to the probability of that message.

[12] As we will show, this choice and further choices that will be made later will lead to a set of signalling strategies $\sigma(m,n)$ yielding a profit asymptotically approaching $\frac{1}{2}$, which is our objective.

For what follows it will be convenient to introduce the following notation:
for $1 \leq i \leq m,\ 1 \leq j \leq n,$

$$y^i_j := \sum_{r=1}^{j} \ell^i_r \text{ and } y^i := y^i_n = \sum_{r=1}^{n} \ell^i_r \tag{19}$$

$$\pi^i_j := y^i_j(1-d^i_j) \text{ and } \pi^i := \pi^i_n = y^i(1-d^i). \tag{20}$$

Thus, y^i is the probability measure of message i and we rewrite (9) as

$$p(i) = y^i = \sum_{j=1}^{n} \ell^i_j = (1-d^i) \quad \text{for} \quad i = 1,\dots,m. \tag{21}$$

Noticing that as the sum of probabilities of all messages is d^m, we have:

$$\sum_{i=1}^{m} y^i = \sum_{i=1}^{m}\sum_{j=1}^{n} \ell^i_j = \sum_{i=1}^{m}(1-d^i) = d^m. \tag{22}$$

Similarly, (y^i_j/y^i) is the H-buyer's (conditional) probability of winning when receiving message i and bidding $d^i_j \cdot v_h$. The corresponding (conditional) profit is

$$u(d^i_j \cdot v_h|v_h,i) = \frac{\pi^i_j}{y^i} v_h \text{ when bidding } d^i_j \cdot v_h \text{ and } u(d^i \cdot v_h|v_h,i) = \frac{\pi^i}{y^i} v_h \text{ when bidding } d^i \cdot v_h. \tag{23}$$

Finally, Equation (18) can now be rewritten as

$$y^i + d^i = 1 \quad \text{for} \quad 1 \leq i \leq m. \tag{24}$$

4. The last necessary condition needed to make $b_h(i) = d^i \cdot v_h$ a best reply of the H-buyer when receiving message i is that this bid is at least as good as bidding at the right-end point of any other segment of the message i that is, $u(d^i_j \cdot v_h|v_h,i) \leq u(d^i \cdot v_h|v_h,i)$ must hold for all $1 \leq i \leq m$ and $1 \leq j \leq n-1$. Here again, let the seller choose the signalling strategy where these inequalities will be satisfied as equalities and using Equation (23) we obtain,

$$\pi^i_j = \pi^i \text{ for all } 1 \leq i \leq m,\ 1 \leq j \leq n-1, \tag{25}$$

which (by Equations (20)) we can write as,

$$y^i_j(1-d^i_j) = y^i(1-d^i) \text{ for all } 1 \leq i \leq m,\ 1 \leq j \leq n-1. \tag{26}$$

This means that when receiving message i, the H-buyer is indifferent between bidding any of the right-end-points of the segments of the message, that is, any $b_h(i) = d^i_j \cdot v_h,\ 1 \leq j \leq n$, is a best reply to the L-buyer's strategy (which is bidding v_ℓ).

Remark 12. *Note that (25) implies that all the intervals of all messages are of positive measure i.e. $\ell^i_j > 0\ \forall i,j$. First, see that for each j we must have at least one i such that $\ell^i_j > 0$ otherwise* *we can dismiss the segment for all messages. By (10), this implies that $d^i_j > d^i_{j-1}$. Finally, if there is an i and j such that $\ell^i_j = 0$, we would have $y^i_j = y^i_{j-1}$ and $d^i_j > d^i_{j-1}$ and hence,*

$$\pi^i_j = y^i_j(1-d^i_j) < y^i_{j-1}(1-d^i_{j-1}) = \pi^i_{j-1},$$

in contradiction to (25).

Remark 13. *It follows from Remark 12 that if $d^i_j \neq d^k_\ell$ if $i \neq k$ or $j \neq \ell$, that is, that no two intervals have the same endpoint.*

Summing up, our construction of the special family of signaling strategies $\sigma(m,n)$ for the seller,

- The strategy $\sigma(m,n)$ is defined by a set of $m \cdot n$ constants (numbers in $[0,1]$), $D = \{d^i_j | i = 1,\dots,m;\ j = 1,\dots,n\}$, from which the vector $L = \{\ell^i_j | i = 1,\dots,m;\ j = 1,\dots,n\}$ is obtained by Equation (10), $Y := \{y^i_j | i = 1,\dots,m;\ j = 1,\dots,n\}$ is obtained by Equation (19), and $\Pi := \{\pi^i_j | i = 1,\dots,m;\ j = 1,\dots,n\}$ is obtained by Equation (20).
- The set of parameters D is chosen so as to satisfy the m equalities in Equations (24) and the $m(n-1)$ equalities in Equations (25).
- This guarantees that the pair of buyers' strategies $E(m,n)$ (described on page 15) will be a Nash equilibrium in the game induced by the signaling strategy $\sigma(m,n)$.
- Among the strategies $\sigma(m,n)$ satisfying the necessary equilibrium conditions, the seller "made" specific choices so as to enable the proof of our main results below.

The existence of such an equilibrium is not immediate since Equations (25) are quadratic equations and the existence of a solution for the $m \cdot n$ equations in (24) and (25) does not follow from standard results. Nevertheless we have:

Theorem 5. *For any $m \geq 1,\ n \geq 1$, there is a solution to Equations (24) and (26).*

Proof. See Appendix A.[13]

Theorem 5 implies that there is a $\sigma(m,n)$ signaling strategy for the seller inducing a game among the buyers with the above described equilibrium.

As we are interested in signaling strategies that maximize the seller revenue, it will be convenient to eliminate some boundary cases in order to simplify the discussions and proofs.

Remark 14. *We can limit our discussion to:*

A. $0 < d^m < 1$*; this implies that the range of values where both values are announced is non-degenerate and the end of the last segment $x^m_n = (c^m_n, d^m_n]$ is strictly between 0 and 1.*

B. $m > 1$ *and* $n > 1$*; that is, there is more than one message and each message is composed of more than one interval.*

To show *A*, we note that $d^m = 0$ means that the seller always reveals the values (v_ℓ, v_h). This strategy yields revenue equal to $1/3$ (which is also the revenue in the benchmark of not revealing any information). As we showed previously, this is not optimal. At the other extreme of $d^m = 1$ is impossible since it implies that the high buyer bids v_h when receiving message m (which happens with positive probability).[14] This cannot happen in equilibrium.

To show *B*, we note that by construction, $m = 1$ implies $n = 1$ and $\sigma(1,1)$ is then the first signaling strategy exhibited in Theorem 2 with revenue $5/12$, which we know can be improved upon by the seller. Similarly, $m > 1$ and $n = 1$ the strategy $\sigma(m,1)$ is the signaling strategy exhibited in Theorem 3 (truncated to the m first segments, i.e., $d^m = \frac{2^m-1}{2^m}$) with revenue less than $4/9$. Again this can be improved upon by the seller (see, for example, Theorem 4). From

[13] We thank Sergiu Hart for whom we owe this proof.

[14] Note that every message has a positive probability, that is, $y^i > 0$, otherwise it can be omitted.

here on we shall assume **A** and **B**.

To explore what revenue can be extracted by the seller using the signaling strategies $\sigma(m,n)$, we recall from the proof of Theorem 1 that given v_h for any signalling strategy the expected seller revenue is bounded by $\frac{3}{4}$ of the surplus which equals $\frac{3}{4}v_h$. In particular,

$$R(\sigma(m,n)|v_h) \leq \frac{3}{4}v_h$$

for any (m,n) and any realized v_h. The following theorem states that this bound cannot be achieved by the strategies $\sigma(m,n)$ for any finite m and n.

Theorem 6. *For any $\sigma(m,n)$, given v_h,*

$$R(\sigma(m,n)|v_h) < \frac{3}{4}v_h.$$

Proof. Assume that $R(\sigma(m,n)|v_h) = \frac{3}{4}v_h$, for some (m,n). This means that the H-buyer's profit is $\frac{1}{4}v_h$, which also equals what he can guarantee by bidding $\frac{1}{2}v_h$ independent of message. We now show that this is impossible.

- For each message i, the potential bid $\frac{1}{2}v_h$ must be contained in the support of the message, i.e., $c^i_j < \frac{1}{2}v_h \leq d^i_j$ for some $1 \leq j \leq n$. To see this, we observe that if for some message i, there is a j where $d^i_{j-1} < \frac{1}{2}v_h \leq c^i_j$ (that is, $\frac{1}{2}v_h$ is the gap between two successive intervals of the message i), then for the H-buyer who received message i, bidding $d^i_{j-1} \cdot v_h$ (the endpoint of the interval at the left of $\frac{1}{2}v_h$) is strictly better than bidding $\frac{1}{2}v_h$ as it has the same probability of winning and pays less when winning. Since every message has a positive probability, this implies that bidding $d^i_{j-1} \cdot v_h$ at message i and $\frac{1}{2}v_h$ otherwise yields a profit strictly greater than $\frac{1}{4}v_h$, in contradiction to $R(\sigma(m,n)|v_h) = \frac{3}{4}v_h$.
- If $c^i_j < \frac{1}{2}v_h < d^i_j$ (i.e., $\frac{1}{2}v_h$ is in the open j-th interval of message i), then by construction (15), bidding $d^i_j \cdot v_h$ is strictly more profitable than bidding $\frac{1}{2}v_h$. Again this implies that that bidding $d^i_j \cdot v_h$ at message i and $\frac{1}{2}v_h$ otherwise yields a profit strictly greater than $\frac{1}{4}v_h$, contradicting the fact that bidding $d^k \cdot v_h$ for every message k (yielding profit of $\frac{1}{4}v_h$) is part of an equilibrium.
- Therefore, for every message i, there is an interval $j(i)$ such that $d^i_{j(i)} = \frac{1}{2}v_h$ in contradiction to Remark 13 since $m > 1$ (by Remark 14). □

Having established the upper bound $\frac{3}{4}E(v_h) = \frac{1}{2}$ for the seller revenue (Theorem 1), we next prove that although this bound cannot be achieved by any signaling strategy $\sigma(m,n)$ (Theorem 6), it is reached asymptotically as (both) m and n tend to infinity.

Theorem 7. *Given $\varepsilon > 0$, there is an integer M such that any signaling strategy $\sigma(m,n)$ with $\min(m,n) > M$ induces a game in which the seller's revenue in the equilibrium is at least $\frac{1}{2} - \varepsilon$.*

Before proving the theorem, in Figures 5, 6, and 7, we present a few numerical computations done with Mathematica, showing the type of strategy $\sigma(m,m)$ and the convergence of the seller's revenue to $\frac{3}{4}v_h$. The choice $m=n$ was done for computational convenience, and $m=n=23$ was the largest size that Mathematica could handle.

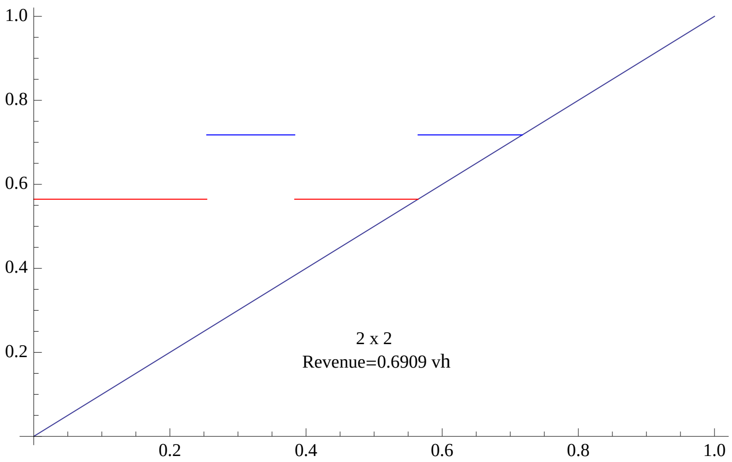

FIGURE 5. *$m=n=2$: Two messages with two intervals each: Revenue* $=0.6909v_h$.

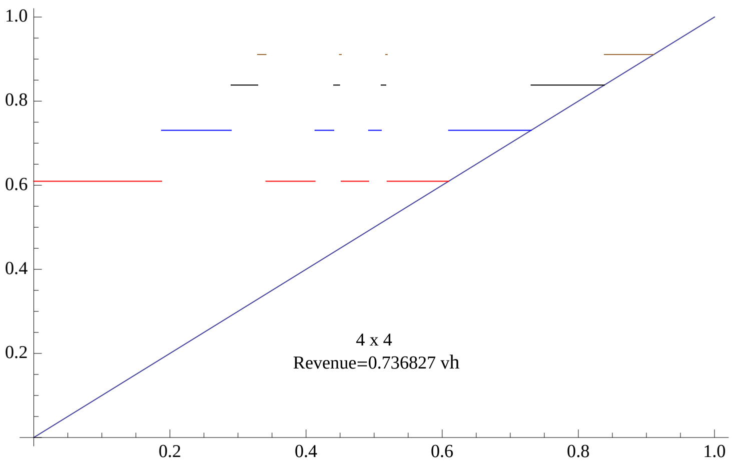

FIGURE 6. *$m=n=4$: Four messages with four intervals each: Revenue* $=0.7368v_h$.

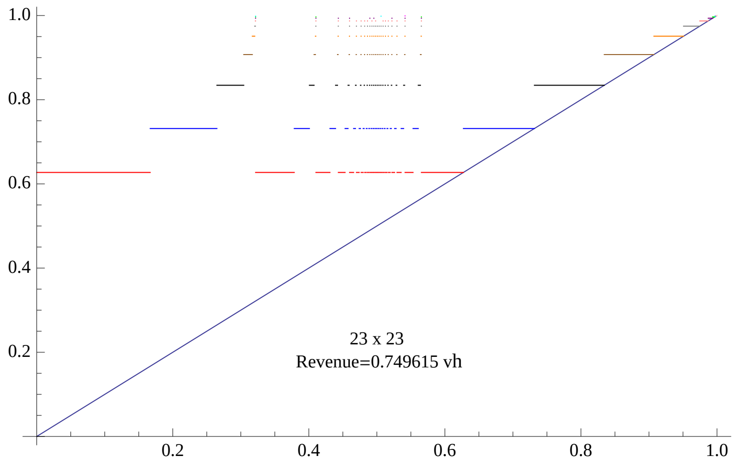

FIGURE 7. $m = n = 23$: *Revenue* $= 0.7496v_h$.

In preparation for the proof of Theorem 7, we derive expressions and bounds in several lemmas that follow. When the seller uses the signaling strategy $\sigma(m,n)$, any value v_h induces a subgame. Since we are interested in the Bayesian-Nash equilibrium of the whole game, we make the analysis for the subgame given v_h and then take the expectation over v_h to compute the profits and revenue in the original (ex-ante) game. First, we will need to find the expressions for the seller revenue and the H-buyer profit in equilibrium when the signaling strategy $\sigma(m,n)$ is used.

Lemma 4. *Given v_h, the expected seller revenue when applying the signaling strategy $\sigma(m,n)$ is*

$$R(\sigma(m,n)|v_h) = \left(\sum_{i=1}^{m} d^i(1-d^i) + \frac{1-(d^m)^2}{2} \right) v_h, \tag{27}$$

and the H-buyer profit is

$$u(\sigma(m,n)|v_h) = \left(\sum_{i=1}^{m} (1-d^i)^2 + \frac{(1-d^m)^2}{2} \right) v_h. \tag{28}$$

Proof. Given v_h, when the buyers follow the Nash equilibrium in the induced (sub) game,

- With probability $p(i) = (1-d^i)$ the message i is announced (by Equation (18)), the H-buyer bids d^i and wins with probability 1. The profit is then $(1-d^i)v_h$ and the seller's revenue is $d^i v_h$.
- With probability $(1-d^m)$ the values (v_ℓ, v_h) are announced, the H-buyer bids v_ℓ and wins yielding him an expected profit (expectation over v_ℓ) equal to $\frac{1-d^m}{2}v_h$ and revenue equal to $\frac{1+d^m}{2}v_h$ to the seller.

This readily yields Equations (27) and (28). □

Next we establish some relationships and bounds on the endpoints d^i of the signals that are satisfied in any equilibrium inducing signaling strategy $\sigma(m,n)$.

Lemma 5. *In any equilibrium signaling strategy $\sigma(m,n)$ with $m>1$ and $n>1$,*

$$1-\frac{1}{i+1}<d^i<1-\frac{1}{2^{i+1}};\;\; i=1,\ldots,m. \tag{29}$$

In particular $\frac{1}{2}<d^1<\frac{3}{4}$ and $1-\frac{1}{m+1}<d^m<1-\frac{1}{2^{m+1}}$.

Proof.

Lower bound: By construction, and using Equations (10) and (18)

$$d^i \equiv d^i_n = \sum_{(k,r)\preceq(i,n)} \ell^k_r = \sum_{r=1}^{n-1}\sum_{k=1}^{m}\ell^k_r + \sum_{k=1}^{i}\ell^k_n \geq \sum_{r=1}^{n}\sum_{k=1}^{i}\ell^k_r = \sum_{k=1}^{i}(1-d^k), \tag{30}$$

where equality holds only for $i=m$. Since by (14) $d^k<d^i$ for $k=1,\ldots,i-1$,

$$\sum_{k=1}^{i}(1-d^k)\geq\sum_{k=1}^{i}(1-d^i)=i(1-d^i),$$

and equality holds only for $i=1$. Therefore, $d^i>i(1-d^i)$, which is $d^i>1-\frac{1}{i+1}$ for all $m>1$ and $i=1,\ldots,m$.

Upper bound: We prove the upper bound in (29) by induction on i:

For $i=1$ we want to show that $d^1<1-\frac{1}{2^2}=\frac{3}{4}$. Intuitively this follows from the fact that the revenue is at least as high as the lower winning bid which is d^1v_h, and the fact that the revenue is less than $\frac{3}{4}v_h$. Formally, by Theorem 6 and Equation (27),

$$\sum_{i=1}^{m} d^i(1-d^i)+\frac{1-(d^m)^2}{2}<\frac{3}{4}, \tag{31}$$

and since the d^i's are strictly increasing (by Equation (14)), we can show that the left hand side of (31) is larger than d^1 with the following steps (using Equations (21) and (22)):

$$\begin{aligned}
\sum_{i=1}^{m} d^i(1-d^i)+\frac{1-(d^m)^2}{2} &> d^1\sum_{i=1}^{m}(1-d^i)+\frac{(1+d^m)(1-d^m)}{2}\\
&> d^1\left(\sum_{i=1}^{m}(1-d^i)+(1-d^m)\right)\\
&= d^1\left(\sum_{i=1}^{m}p(i)+(1-d^m)\right)\\
&= d^1\left(d^m+(1-d^m)\right)=d^1,
\end{aligned}$$

establishing that $d^1 < \frac{3}{4}$ and hence $d^i < 1 - \frac{1}{2^{i+1}}$ for $i = 1$.

For the inductive step we need the following:

$$d^{i+1} < \frac{1+d^i}{2}; \;\; i = 1, \ldots, m. \tag{32}$$

Indeed by construction of $\sigma(m,n)$ (Equation (10)), we have

$$d^{i+1} \equiv d_n^{i+1} = d_n^i + \ell_n^{i+1} = d^i + \ell_n^{i+1}$$

and by (18), $\ell_n^{i+1} < 1 - d^{i+1}$ (since $n > 1$), we get

$$d^{i+1} = d^i + \ell_n^{i+1} < d^i + (1 - d^{i+1}), \;\text{ and hence }\; d^{i+1} < \frac{1+d^i}{2}.$$

The inductive step follows now readily from (32). Assuming that $d^i < 1 - \frac{1}{2^{i+1}}$ then,

$$d^{i+1} < \frac{1+d^i}{2} < \frac{1}{2}[1 + (1 - \frac{1}{2^{i+1}})] = 1 - \frac{1}{2^{i+2}}.$$

This concludes the proof of Lemma (5). □

Remark 15. *Interestingly, for d^m (the endpoint of all messages), an upper bound smaller than $d^m < 1 - \frac{1}{2^{m+1}}$ can be proved, namely, $d^m < 1 - \frac{1}{2^m}$. As this is not used for our results, we do not present the proof.*

Lemma 6. *For any $m > 1$, in the signaling strategy $\sigma(m,n)$, the length of the first segment of the first message, ℓ_1^1, satisfies $\ell_1^1(1 - \ell_1^1) \geq \frac{1}{16}$. As a result, we have $\ell_1^1 > \frac{1}{16}$.*

Proof. By construction of $\sigma(m,n)$, the H-buyer, when receiving message 1, is indifferent between bidding $d^1 \cdot v_h$ yielding a profit $(1 - d^1)v_h$ with probability 1, or bidding $d_1^1 \cdot v_h = \ell_1^1 \cdot v_h$ yielding a profit $(1 - \ell_1^1)v_h$ with probability $\frac{\ell_1^1}{\sum_{j=1}^n \ell_j^1}$. Thus, by (18),

$$(1 - d^1)v_h = \frac{\ell_1^1}{\sum_{j=1}^n \ell_j^1}(1 - \ell_1^1)v_h = \frac{\ell_1^1}{(1 - d^1)}(1 - \ell_1^1)v_h \;\text{ which is }\; \ell_1^1(1 - \ell_1^1) = (1 - d^1)^2,$$

and thus, since $d^1 < \frac{3}{4}$ (by Lemma 5), we get

$$\ell_1^1(1 - \ell_1^1) = (1 - d^1)^2 > (1 - \frac{3}{4})^2 = \frac{1}{16}. \quad \square \tag{33}$$

We are now ready to prove our main theorem, Theorem 7 (page 21).

Proof of Theorem 7

When the seller uses the signaling strategy $\sigma(m,n)$, any value v_h induces a subgame. Since we are interested in subgame-perfect equilibrium, we make the analysis for the subgame given v_h

and then take the expectation over v_h to compute the profits and revenue in the original (ex-ante) game.

By Theorem 6 we know that $R(\sigma(m,n)|v_h) < \frac{3}{4}v_h$ for all (m,n) and we want to show that for $\varepsilon > 0$ there exist an $M(\varepsilon)$ such that $R(\sigma(m,n)|v_h) > (\frac{3}{4} - \varepsilon)v_h$ for all (m,n) such that $\min(m,n) > M(\varepsilon)$. As the total surplus (given v_h) is v_h, we shall show that $u(\sigma(m,n)|v_h) \leq (\frac{1}{4} + \varepsilon)v_h$ for all (m,n) such that $\min(m,n) > M(\varepsilon)$. The proof proceeds along the following lines:

- As $\min(m,n) \to \infty$, there will be a point $x \in (0,1)$ such that each message has a segment whose endpoint is arbitrarily close to x.
- Since when receiving message i, bidding the endpoint of any segment of that message is a best reply for the H-buyer, bidding $x \cdot v_h$ for all messages is close to best reply, with profit close to the equilibrium profit.
- Therefore, the H-buyer's profit in the game induced by $\sigma(m,n)$ (with large enough m) is within ε of his profit from ignoring the messages and always bidding $x \cdot v_h$. This strategy yields a profit of $x(1-x) \cdot v_h$.
- Since $x(1-x) \leq \frac{1}{4}$, the profit of the H-buyer is at most $x(1-x)v_h + \varepsilon$, hence the seller's revenue is at least $\frac{3}{4}v_h - \varepsilon$.

Formally, we proceed as follows:

Step 1. Given $\varepsilon > 0$, choose $M(\varepsilon) = \max\{M_1, M_2\}$ such that:

i) M_1 is large enough such that for $n > M_1$ there will be a segment of the first message, say segment r, of length $\ell_r^1 < \delta$ for a fixed $\delta < \min\{\frac{\varepsilon}{2}, \frac{1}{16}\}$. (This is possible since $\sum_{j=1}^{n} \ell_j^1 < 1$.)

ii) M_2 is large enough such that for $m > M_2$ we have $\frac{(1-d^m)^2}{2} < \frac{\varepsilon}{2}$ (using Lemma 5, page 24).

From now on consider signaling strategies $\sigma(m,n)$ with $\min(m,n) > M(\varepsilon)$.

Step 2. Fixing such an r from Step 1, since by Lemma 6 (page 25), $\ell_1^1 > \frac{1}{16} > \delta$, it must be that $r \geq 2$ (this is not the first segment of message 1).

Step 3. The difference $d_r^1 - d_{r-1}^1$ (between the r-th right-end point and $(r-1)$-th right-end point in message 1) satisfies:

$$d_r^1 - d_{r-1}^1 < 16\delta. \tag{34}$$

This follows since when receiving message 1, the H-buyer is indifferent between bidding $d_r^1 \cdot v_h$ or $d_{r-1}^1 \cdot v_h$, that is:

$$\frac{\sum_{j=1}^{r-1} \ell_j^1}{1-d^1}(1-d_{r-1}^1) = \frac{\sum_{j=1}^{r} \ell_j^1}{1-d^1}(1-d_r^1),$$

and consequently,

$$\begin{aligned}
(1-d^1_{r-1})\sum_{j=1}^{r-1}\ell_{1j} &= (1-d^1_r)(\sum_{j=1}^{r-1}\ell^1_j+\ell^1_r) \\
(d^1_r-d^1_{r-1})\sum_{j=1}^{r-1}\ell^1_j &= \ell^1_r(1-d^1_r) \\
(d^1_r-d^1_{r-1}) &= \frac{\ell^1_r(1-d^1_r)}{\sum_{j=1}^{r-1}\ell^1_j} \\
&< \frac{\ell^1_r}{\ell^1_1} < 16\ell^1_r < 16\delta.
\end{aligned}$$

Step 4. By construction, the $(r-1)$-th segments of all messages $i \geq 1$, are in the interval $[d^1_{r-1}, d^1_r]$ therefore, $d^1_{r-1} < d^i_{r-1} < d^1_r$ for all $i = 2, \ldots, m$ and by (34),

$$d^1_r - d^i_{r-1} < 16\delta \ \text{ for all } \ i = 1, \ldots, m. \tag{35}$$

Step 5. Proceeding to estimate the H-buyers' profit, by Equation (28),

$$\begin{aligned}
u(\sigma(m,n)|v_h) &= \left(\sum_{i=1}^{m} y^i(1-d^i)+\frac{(1-d^m)^2}{2}\right)v_h \\
\text{by (26) and } m>M_2 \quad &= \left(y^1_r(1-d^1_r)+\sum_{i=2}^{m} y^i_{r-1}(1-d^i_{r-1})+\frac{\varepsilon}{2}\right)v_h \\
\text{by Equation (35)} \quad &\leq \left(y^1_r(1-d^1_r)+\sum_{i=2}^{m} y^i_{r-1}(1-d^1_r+16\delta)+\frac{\varepsilon}{2}\right)v_h \\
&= \left((y^1_r+\sum_{i=2}^{m} y^i_{r-1})(1-d^1_r)+16\delta\sum_{i=2}^{m} y^i_{r-1}+\frac{\varepsilon}{2}\right)v_h \\
\text{by Equation (19)} \quad &= \left((\sum_{(k,j)\preceq(1,r)}\ell^k_j)(1-d^1_r)+16\delta\sum_{i=2}^{m} y^i_{r-1}+\frac{\varepsilon}{2}\right)v_h \\
\text{by Equation (10)} \quad &< \left(d^1_r(1-d^1_r)+16\delta d^1_r+\frac{\varepsilon}{2}\right)v_h.
\end{aligned}$$

Finally, since $d^1_r < 1$, $16\delta < \frac{\varepsilon}{2}$ and $x(1-x) \leq \frac{1}{4}$ for all $x \in [0,1]$ (in particular, $x = d^1_r$), we get:

$$u(\sigma(m,n)|v_h) < \left(d^1_r(1-d^1_r)+16\delta+\frac{\varepsilon}{2}\right)v_h < \left(\frac{1}{4}+\varepsilon\right)v_h,$$

completing the proof of Theorem 7. □

Remark 16. *Equation (35) states that all messages have a segment with an endpoint close to d^1_r (within 16δ), establishing the first bullet point in the plan of the proof on page 26 with $x = d^1_r$.*

Remark 17. *Since we know that the H-buyer can guarantee $\frac{1}{4}v_h$, it follows that the corresponding points $d_r^1(m,n)$ in our proof (as function of m and n) must satisfy: $\lim_{m\to\infty,n\to\infty} d_r^1(m,n) = \frac{1}{2}$, as it is visible in the figures of our numerical computations. Recall that ignoring the messages and bidding $\frac{1}{2}v$ is the Maxmin strategy for each buyer guaranteeing $\frac{1}{8}E(v_h) = \frac{1}{12}$ (see Lemma 1 on page 6).*

In the following we prove that the bound $\frac{3}{4}v_h$ for the revenue is achieved asymptotically by $\sigma(m,n)$ only when both m and n go to infinity. First we prove that $m \to \infty$ is necessary:

Theorem 8. *For any finite m there is $\varepsilon_m > 0$ such that $R(\sigma(m,n)) \leq (\frac{3}{4} - \varepsilon_m)v_h$ for all n, that is, the revenue $R(\sigma(m,n))$ is bounded away from $(3/4)v_h$ uniformly in n.*

Proof. See Appendix.

Now we prove the counterpart of Theorem 8 namely that $n \to \infty$ is necessary:

Theorem 9. *For any finite n there is $\varepsilon_n > 0$ such that $R(\sigma(m,n)) \leq (\frac{3}{4} - \varepsilon_n)v_h$ for all $m \geq 2$, that is, the revenue $R(\sigma(m,n))$ is bounded away from $(3/4)v_h$ uniformly in m.*

Proof. See Appendix.

5 Outline of generalizations

Our results can be extend in two directions:

1. The distribution of values can be more general, that is, non-uniform distributions and asymmetric distributions between the buyers.
2. The results hold for more than two buyers.

For the first generalization, we need to adjust the definition of ℓ_j^i (on page 15) to $\ell_j^i = P(\frac{v_\ell}{v_h} \in x_j^i | v_h, v_\ell \leq v_h)$. This is the conditional probability, given v_h and $v_\ell \leq v_h$, that $\frac{v_\ell}{v_h}$ is in the interval x_j^i.

Let F_{ℓ,v_h} be the cumulative conditional distribution of $\frac{v_\ell}{v_h}$ given v_h and given that v_h is the high value, that is, $F_{\ell,v_h}(x) = P(\frac{v_\ell}{v_h} \leq x | v_h, v_\ell \leq v_h)$. We can now generalize Equation (10):

$$d_j^i = F_{\ell,v_h}^{-1}\Big(\sum_{(k,r)\preceq(i,j)} \ell_r^k\Big) \quad \text{for } i = 1,\ldots,m;\ j = 1,\ldots,n. \tag{36}$$

With these modifications Equations (24) and (26) are used to determine the equilibrium in the same manner. Note that unlike the uniform case, the signalling strategy of the seller depends upon the realization of v_h.

For the generalization to more than two buyers, we note that our equilibrium of interest is where the buyers without the highest value bid their value. Thus, if N is the set of buyers and v_j is the value of buyer j; $j \in N$, we let

$$v_h = \max\{v_j | i \in N\},$$

$$v_\ell = \max\{v_j | v_j < v_h\}.$$

As the conditional distribution of v_ℓ given v_h may not be uniform (as with the case for more than two buyers when values of each buyer are from the uniform distribution), the first generalization is called for to make the appropriate modifications.

6 Conclusion

In this paper, we consider a first-price auction where a seller might have information about the buyers' valuations and can strategically communicate with the buyers via public messages with the objective to increase revenue. The natural way to model the informed seller is as a player in a game with n buyers (and $n+1$ players). By doing so, we are able to make use of the maximin concept to find the maximum possible revenue. We provide a classification of commitment power of the seller and show the importance of commitment power to results. We find that the value of information is related not only to how much information a seller has, but to his commitment power.

While we solved for this is a small part of a more general model of auctions, where a seller might not know the exact values of the buyers and only on a less refined partition. There are many interesting questions left by following this new direction. For instance, is revenue monotonic in terms of increasingly refined partitions of seller information. Furthermore, it is of interest to find general conditions when additional commitment power over verifiability commitment will be beneficial to the sender of information.

We note that having a multi-stage auction mechanism (such as in Perry, Wolfstetter, and Zamir (2000)) where the seller may learn information in the first period and release it in the second cannot result in the equilibrium in our paper (due to revenue equivalence). However, information in multi-stage auctions could be an issue when in a further stage there is a possibility of resale (see Zheng (2002), Lebrun (2010)) or an additional auction in the case of no sale (Skreta 2015). Another interesting direction to take when there is a possibility for the seller to acquire information before deciding to release it (in a similar vein to Kim and Koh (2020)).

7 Appendix

7.1 Proof of Theorem 5

We want to prove the existence of solution for the mxn vector of strictly positive variables $\ell := \{\ell^i_j | i = 1,\dots,m;\ j = 1,\dots,n\}$ satisfying the m linear equations (24), on page 18:

$$y^i + d^i = 1 \text{ for } 1 \le i \le m, \tag{37}$$

and the $m(n-1)$ quadratic equations (26):

$$y^i_j(1-d^i_j) = y^i(1-d^i) \ \text{ for } 1 \le i \le m,\ 1 \le j \le n-1. \tag{38}$$

Replacing y^i_j, d^i_j, y^i, d^i in terms of ℓ^i_j using Equations (10), (11) and (19) (on pages 16 and 18), yields:

$$\sum_{r=1}^{n} \ell^i_r + \sum_{(k,r)\preceq(i,n)} \ell^k_r = 1 \text{ for } 1 \le i \le m, \tag{39}$$

$$(\sum_{r=1}^{j} \ell^i_r)(1 - \sum_{(k,r)\preceq(i,j)} \ell^k_r) = (\sum_{r=1}^{n} \ell^i_r)(1 - \sum_{(k,r)\preceq(i,n)} \ell^k_r) \text{ for } 1 \le i \le m,\ 1 \le j \le n-1. \tag{40}$$

For the proof we will use the following variant of Brouwer's fixed point theorem by Hartman and Stampacchia called the Variational Inequality (see Section 8.1 of Border (1989), page 41).

Theorem 10. *Let $C \subset \mathbb{R}^m$ be a nonempty compact and convex set and let $f : C \to \mathbb{R}^m$ be a continuous function. Then there exists $y \in C$ such that*

$$(c-y)\cdot f(y) \le 0$$

for every $c \in C$.

In preparation for the proof of Theorem 5 we prove several lemmas.

Lemma 7. *By the linear equations (39), the $m(n-1)$ variables ℓ^i_j for all i and all $j \ge 2$ uniquely determine ℓ^i_1 for all i (that is, the first interval of each message).*

Proof. Given ℓ^i_j for $j \ge 2$, equations (39) are m linear equations in $\ell_1 := (\ell^i_1)^m_{i=1}$ (viewed a column vector). The coefficient of ℓ^i_1 is 2 in the i-th equation and 1 in all other $m-1$ equations for $k \ne i$ therefore,

$$A\ell_1 = b(\tilde{\ell}) \quad \text{with} \quad A = I + E, \tag{41}$$

where I is the unit matrix, E is the matrix of all ones, and $b(\tilde{\ell}) := (b^i)^m_{i=1}$ is a column vector determined by $\tilde{\ell} := (\ell^i_j)^{1\le i\le m}_{j\ge 2}$.

The inverse of the matrix A is,

$$A^{-1} = I - \frac{1}{m+1}E; \tag{42}$$

i.e., $m/(m+1)$ on the diagonal, and $-1/(m+1)$ off the diagonal. □

Let $L := \{\ell \equiv (\ell^i_j)_{1\le i\le m, 1\le j\le n} \in \mathbb{R}^{mn} : \ell \ge 0 \text{ and } \ell \text{ satisfies Equations (39)}\}$, and let $\tilde{L}$ be its projection on $\mathbb{R}^{m(n-1)}$ obtained by dropping the first m coordinates ℓ^i_1; i.e., $\tilde{L} := \{\tilde{\ell} \equiv (\ell^i_j)_{1\le i\le m, 2\le j\le n} : \ell \in L\}$. As seen above, given $\tilde{\ell} \in \tilde{L}$ there is a unique $\ell \in L$ corresponding to it (determined by Equations (41)). $L \ne \emptyset$ since the vector $\ell_0 := \{\ell^i_1 = \frac{1}{m+1},\ i = 1,\dots,m, \text{ and } \ell^i_j = 0 \text{ for } j \ne 1\}$, is in L (can be directly verified to satisfy Equations (39)). Hence, the set $\tilde{L}$ is nonempty (includes the vector $\ell^i_j = 0$ for all i and all $j \ge 2$), convex compact polytope in $\mathbb{R}^{m(n-1)}$.

Recalling the definition of Equation (20) on page 18,

$$\pi^i_j := y^i_j(1-d^i_j) \text{ and } \pi^i := \pi^i_n = y^i(1-d^i),$$

we present the following lemma which is a corollary of Lemma 5.

Lemma 8. *If $\ell \in L$ then $d^i_j < 1$ for all i, j, and consequently $\pi^i_n > 0$ for all i.*

Proof. We note that $\pi^i_n = (1-d^i_n)^2 > 0$ (the equality by (37)). $\square$

Lemma 9. *Let $(\ell_1, \tilde{\ell}) \in L$.*

(i) For every i and j where $2 \le j \le n-1$, if we have $\tilde{\ell}^i_j > 0$, then $\tilde{\ell} - \varepsilon \mathbf{1}^i_j \in \tilde{L}$ for small enough $\varepsilon > 0$.

(ii) For every i, if $\ell^i_n > 0$ and $\ell^k_1 > 0$ for all $k < i$ then $\tilde{\ell} - \varepsilon \mathbf{1}^i_n \in \tilde{L}$ for small enough $\varepsilon > 0$.

(iii) If $\ell^i_1 > 0$ for all i, then for every i and every j, where $2 \le j \le n$; $\tilde{\ell} + \varepsilon \mathbf{1}^i_j \in \tilde{L}$ for small enough $\varepsilon > 0$.

Proof. (i) In each of the equations in (39), ℓ^i_1 and ℓ^i_j (for $2 \le j \le n-1$) appear with the same coefficient (namely, 1 if $k \ne i$ and 2 if $k = i$), and so if $\tilde{\ell}$ is the projection of $\ell \in L$ then $\ell_\varepsilon := \ell - \varepsilon \mathbf{1}^i_j + \varepsilon \mathbf{1}^i_1 \in L$; and its projection which is $\tilde{\ell}_\varepsilon = \tilde{\ell} - \varepsilon \mathbf{1}^i_j$ is in $\tilde{L}$.

(ii) In the k-th equation of (39), ℓ^i_n appears with coefficient 0 if $k < i$, with coefficient 2 if $k = i$, and with coefficient 1 if $k > i$. Therefore if we let $\tilde{c} = \tilde{\ell} - \varepsilon \mathbf{1}^i_n$, the right hand side in equation (41) satisfies,

$$b(\tilde{c}) = b(\tilde{\ell}) + v \quad \text{with} \quad v := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 2\varepsilon \\ \varepsilon \\ \vdots \\ \varepsilon \end{bmatrix}, \tag{43}$$

where 2ε is in coordinate i.

Let $u = (u^1, ..., u^m)$ be given by $u = A^{-1}v$, then u^k is easily seen by (42) to be negative for $k < i$ and nonnegative for $i \ge k$. Define now $c = (c_1, \tilde{c})$ by $c^i_j := \ell^i_j - \varepsilon$, $c_1 = \ell_1 + u$ i.e., $c^k_1 := \ell^k_1 + u^k$ for all k, and $c^{i'}_{j'} := \ell^{i'}_{j'}$ for all other coordinates (i.e., $(i', j') \ne (i, j)$ and $j' > 1$). We claim that for $\varepsilon > 0$ small enough, $c \in L$. First, for $\varepsilon > 0$ sufficiently small, $c^i_j \ge 0$ for all (i, j) because the coordinates that decrease, namely, ℓ^k_1 for $k < i$ and ℓ^i_n, are all positive. To show that $c \in L$ we have to show that for $\varepsilon > 0$ sufficiently small, c satisfy equation (41) which is true because $u = Av$ and ℓ satisfy equation (41),

$$Ac_1 = A(\ell_1 + u) = A\ell_1 + Au = b(\tilde{\ell}) + v = b(\tilde{c}).$$

Thus, $c \in L$ and $\tilde{c} = \tilde{\ell} - \varepsilon \mathbf{1}^i_n \in \tilde{L}$.

(iii) Proceed as in (ii); because all the ℓ^k_1 are all positive, the signs of the resulting u^k do not matter. $\square$

We turn now to the proof of Theorem 5:

Define the function $f = (f^i_j)_{1 \le i \le m, 2 \le j \le n} : \tilde{L} \to \mathbb{R}^{m(n-1)}$ by

$$f^i_j(\tilde{\ell}) := \pi^i_{j-1} - \pi^i_j$$

for every i and every $j \geq 2$.

Equations (38) imply that a high-value buyer is indifferent between bidding at the top of any of the segments in the message that he receives. The function f is the difference in profits between consecutive segments of a message. Thus, the existence of a zero point, that is, $\tilde{\ell} \in \tilde{L}$ such that $f(\tilde{\ell}) = 0$ implies the existence of a solution to Equations (38).

The function f is clearly continuous, and so, by Theorem 10, there is $\tilde{\ell} \in \tilde{L}$ such that

$$\tilde{\ell} \cdot f(\tilde{\ell}) \geq \tilde{c} \cdot f(\tilde{\ell}) \text{ for every } \tilde{c} \in \tilde{L}. \quad (44)$$

We will show that (44) implies that $f(\tilde{\ell}) = 0$.

First, we claim that $f(\tilde{\ell}) \geq 0$. Otherwise, let (i, j) be the smallest in the order $\preceq$ such that $f^i_j(\tilde{\ell}) < 0$. This implies that $\ell^i_j > 0$, because if $\ell^i_j = 0$ then $y^i_j = y^i_{j-1}$ and thus $f^i_j(\tilde{\ell}) = \pi^i_{j-1} - \pi^i_j = y^i_{j-1}(d^i_j - d^i_{j-1}) \geq 0$. If $j \leq n-1$ then Lemma 9 (i) implies that $\tilde{c} := \tilde{\ell} - \varepsilon \mathbf{1}^i_j \in \tilde{L}$ for $\varepsilon > 0$ small enough, and so $(\tilde{\ell} - \tilde{c}) \cdot f(\tilde{\ell}) = \varepsilon f^i_j(\tilde{\ell}) < 0$, contradicting (44). If $j = n$ then for all $k < i$ we have $\pi^k_1 \geq \pi^k_2 \geq ... \geq \pi^k_n$ (because $f^k_2(\tilde{\ell}), ..., f^k_n(\tilde{\ell}) \geq 0$, as all these come before (i, j)); now $\pi^k_n > 0$ by Lemma 8, and so $\pi^k_1 > 0$, which implies that $\ell^k_1 = y^k_1 > 0$; finally, Lemma 9 (ii) implies that $\tilde{c} := \tilde{\ell} - \varepsilon \mathbf{1}^i_n \in \tilde{L}$ for $\varepsilon > 0$ small enough, and so $(\tilde{\ell} - \tilde{c}) \cdot f(\tilde{\ell}) = \varepsilon f^i_n(\tilde{\ell}) < 0$, again contradicting (44).

Therefore we have shown for all i that

$$\pi^i_1 \geq \pi^i_2 \geq ... \geq \pi^i_n > 0$$

(the positivity is by Lemma 8), which implies that $\ell^i_1 = y^i_1 > 0$ for all i.

For every i and every $j \geq 2$ Lemma 9 (iii) yields $\tilde{c} := \tilde{\ell} + \varepsilon \mathbf{1}^i_j \in \tilde{C}$ for $\varepsilon > 0$ small enough, and so $0 \leq (\tilde{\ell} - \tilde{c}) \cdot f(\tilde{\ell}) = -\varepsilon f^i_j(\tilde{\ell})$ by (44), which shows that $f(\tilde{\ell}) \leq 0$.

Altogether $f(\tilde{\ell}) = 0$, which implies that for each i all the π^i_j are equal.

Finally, $\ell^i_j > 0$ for all (i, j), since otherwise take the first (i, j) in the order $\preceq$ where $\ell^i_j = 0$, then $j \geq 2$ (we saw above that $\ell^i_1 > 0$ for all i) and $\pi^i_j < \pi^i_{j-1}$ (we have $d^i_j > d^i_{j-1}$ and $y^i_j = y^i_{j-1} > 0$ because (i, j) is the smallest with $\ell^i_j = 0$), which contradicts $f^i_j(\tilde{\ell}) = 0$. This completes the proof of Theorem 5 □

7.2 Proof of Theorem 8

The proof of Theorem 8 is a consequence of the following observations:

- Given v_h, for all $n \geq 1$ and $m \geq 2$,

$$R(\sigma(m,n)) \leq \frac{1}{2}[\frac{3}{2} - d^m + (d^m)^2]v_h. \quad (45)$$

  To see that we observe that by bidding $d^m/2 \cdot v_h$ whenever he receives one of the m signals, the H-buyer wins with probability $\frac{1}{2}$ a profit of $(1 - d^m/2)v_h$. Thus guaranteeing $\frac{1}{2}(1 -$

$\frac{d^m}{2})v_h$ when $v_\ell \le d^m v_h$ (probability d^m) and $(v_h - v_\ell)$ when $d^m v_h < v_\ell \le v_h$ (probability $1-d^m$). Overall, taking conditional expectation over v_ℓ, the H-buyer can guarantee

$$\pi_H(\sigma(m,n)) \ge \frac{d^m}{2}(1-\frac{d^m}{2})v_h + (1-d^m)\frac{1+d^m}{2}v_h = \frac{1}{2}[\frac{1}{2} - d^m + (d^m)^2]v_h.$$

Hence, the total revenue of the seller is at most:

$$R(\sigma(m,n)) \le v_h - \frac{1}{2}[\frac{1}{2} - d^m + (d^m)^2]v_h = \frac{1}{2}[\frac{3}{2} - d^m + (d^m)^2]v_h. \quad (46)$$

- The right hand side of (46) is strictly increasing in d^m for $d^m > \frac{1}{2}$ (which is the case by (29)) and is equal to $(3/4)v_h$ at $d^m = 1$.
- By (29) again, $d^m < 1 - \frac{1}{2^{m+1}} < 1$ for all $n \ge 1$. Hence, we can then set $\varepsilon_m = \frac{2^{m+1}-1}{2^{2m+3}}$ using (46), completing the proof of Theorem 8. □

7.3 Proof of Theorem 9

For the proof of this theorem we prove first that for any finite n, the measure of each segment in the first two messages is bounded away from zero.

Lemma 10. *Given n let $\delta_n := (\frac{1}{16}\cdot\frac{1}{64})^n = 2^{-10n}$, then*

$$\ell^i_j > \delta_n \quad \textit{for all} \quad i = 1,2 \;\; \textit{and} \;\; j = 1,\dots,n. \quad (47)$$

Proof. First observe that by Lemma 5 (page 24), $d^1 > \frac{3}{4}$ and $d^2 > \frac{7}{8}$ and by Equations (24) and (26) for $i = 1,2$ and $j = 1$ we have (recalling that all segments are of positive measure, by Remark 12),

$$\ell^1_1 > \ell^1_1(1-d^1_1) = (1-d^1)^2 > (\frac{1}{4})^2 = \frac{1}{16}.$$

Similarly

$$\ell^2_1 > \ell^2_1(1-d^2_1) = (1-d^2)^2 > (\frac{1}{8})^2 = \frac{1}{64},$$

and so,

$$\ell^1_1\ell^2_1 > \frac{1}{16}\cdot\frac{1}{64} = 2^{-10}. \quad (48)$$

Next, using Equations (26) again we have

$$y^1_{j+1}(1-d^1_{j+1}) = y^1_j(1-d^1_j) \;\; \text{and} \;\; y^2_{j+1}(1-d^2_{j+1}) = y^2_j(1-d^2_j),$$

which, since $y^i_{j+1} = y^i_j + \ell^i_{j+1}$, can be written as

$$\ell^1_{j+1}(1-d^1_{j+1}) = y^1_j(d^1_{j+1} - d^1_j) \;\; \text{and} \;\; \ell^2_{j+1}(1-d^2_{j+1}) = y^2_j(d^2_{j+1} - d^2_j),$$

respectively. Now by the construction of $\sigma(m,n)$, $d^1_{j+1} - d^1_j \ge \ell^2_j$ and $d^2_{j+1} - d^2_j \ge \ell^1_{j+1}$, and since $(1-d^i_{j+1}) < 1$ and $y^i_j \ge \ell^i_1$, we get

$$\ell^1_{j+1} > \ell^1_{j+1}(1-d^1_{j+1}) \ge \ell^1_1\ell^2_j \;\; \text{and similarly} \;\; \ell^2_{j+1} > \ell^2_{j+1}(1-d^2_{j+1}) \ge \ell^2_1\ell^1_{j+1},$$

and so, for $j = 1,\ldots,n$,

$$\ell^1_{j+1} > \ell^1_1 \ell^2_j \quad \text{and} \quad \ell^2_{j+1} > \ell^2_1 \ell^1_{j+1}. \tag{49}$$

Finally from Equation (49) we get

$$\ell^1_j > (\ell^1_1\ell^2_1)^{j-1}\ell^1_1 \ \text{ and } \ \ell^2_j > (\ell^1_1\ell^2_1)^j; \ j = 1,\ldots,n,$$

which, using Equation (48), yields Equations (47). □

We can now prove of Theorem 9.

We shall prove that given v_h and given n and any signaling strategy $\sigma(m,n)$, the H-buyer has a bidding strategy that guarantees a profit $\pi_H(\sigma(m,n)) \geq (\frac{1}{4} + \varepsilon_n)v_h$, and hence $R(\sigma(m,n)) \leq (\frac{3}{4} - \varepsilon_n)v_h$. Recalling that by bidding always (i.e., after any message received from the seller including (v_1,v_2)) $\frac{1}{2}v_h$ the H-buyer guarantees a profit of $\frac{1}{4}v_h$ (Theorem 1), we shall show that for any m, bidding differently in message 1 or 2 (or both) and bidding $\frac{1}{2}v_h$ after all messages $i \geq 3$, he can increase his profit by at least $\frac{3}{4}\delta_n^2 v_h$ which proves the theorem letting $\varepsilon_n = \frac{3}{4}\delta_n^2$.

To exhibit such a strategy, given the signaling strategy $\sigma(m,n)$ recall that by construction, any j-th segment of message 1 (of length ℓ^1_j) is followed by the j-th segment of message 2 (of length ℓ^2_j) that is: $d^1_j = d^2_j - \ell^2_j$ for all $j = 1,\ldots,n$. Consider the following cases:

(a) If $\frac{1}{2} \leq d^1_1 = \ell^1_1$, then bidding $\frac{1}{2}v_h$ after message 2 yields zero probability of winning (since all the support of message 2 is greater than $\frac{1}{2}$). By bidding $d^2 \cdot v_h$, the H-buyer wins $(1-d^2)\cdot v_h$ with probability $y^2 = (1-d^2)$ (by (24)) and thus increases his profit by at least $(1-d^2)^2 \cdot v_h > (\frac{1}{8})^2 \cdot v_h > \delta_n^2 \cdot v_h$ (by (29)).

If it is not case (a), then $d^1_1 < \frac{1}{2}$. Since by (29), $d^1 = d^1_n > \frac{1}{2}$, let $j_* = \min\{j | d^1_j > \frac{1}{2}\}$.

(b) If $d^1_{j_*-1} + \ell^2_{j_*-1} = d^2_{j_*-1} \leq \frac{1}{2}$ and $d^1_{j_*} - \ell^1_{j_*} \geq \frac{1}{2}$ (i.e., $\frac{1}{2}$ is between $d^2_{j_*-1}$ and $d^1_{j_*} - \ell^1_{j_*}$ then if at message 1 the H-buyer bids $d^1_{j_*-1}v_h$ instead of $\frac{1}{2}v_h$, he will have the same probability of winning $y^1_{j_*} \geq \ell^1_1 > \delta_n$, and paying $(\frac{1}{2} - d^1_{j_*-1})\cdot v_h \geq \ell^2_{j_*}\cdot v_h > \delta_n \cdot v_h$ less when winning (for message 1), and thus increasing profit by at least $\delta_n^2 \cdot v_h$.

(c) If $d^2_{j_*-1} - \ell^2_{j_*-1} \leq \frac{1}{2} < d^2_{j_*-1}$ (i.e., $\frac{1}{2}$ is in the (j_*-1)-th segment of message 2), let $x = d^2_{j_*-1} - \frac{1}{2}$ then $0 \leq x \leq \ell^2_{j_*-1}$ and if at message 2 the H-buyer bids $d^2_{j_*-1}\cdot v_h$ instead of $\frac{1}{2}\cdot v_h$, he increases his profit by:

$$[y^2_{j_*-1}(1-d^2_{j_*-1}) - (y^2_{j_*-1} - x)(1 - d^2_{j_*-1} + x)]v_h = [x(1 - d^2_{j_*-1} - y^2_{j_*-1} + x)]v_h \geq x^2 v_h,$$

where the last inequality is by using (17) and (19) (page 17). If in addition, at message 1 the H-buyer bids $d^1_{j_*-1}\cdot v_h = (\frac{1}{2} - (\ell^2_{j_*-1} - x))\cdot v_h$ instead of $\frac{1}{2}\cdot v_h$, his probability of winning is unchanged but he pays $(\ell^2_{j_*-1} - x)v_h$ less when winning (for message 1) and thus his profit is increased by $(\ell^2_{j_*-1} - x)y^1_{j_*-1}v_h > (\ell^2_{j_*-1} - x)\ell^1_1 v_h > (\delta_n - x)\delta_n v_h$ (by Lemma 10).

Altogether, the deviations from the $\frac{1}{2}\cdot v_h$ bid in messages 1 and 2 yield an increase of profit greater than $[x^2 + (\delta_n - x)\delta_n]v_h$ which attains its minimum at $x = \delta_n/2$ where it is equal to $\frac{3}{4}\delta_n^2 v_h$.

(d) Finally if $d^1_{j_*} - \ell^1_{j_*} \le \frac{1}{2} < d^1_{j_*}$ (i.e., $\frac{1}{2}$ is in the j_*-th segment of message 1), let $x = d^1_{j_*} - \frac{1}{2}$ then $0 \le x \le \ell^1_{j_*}$ and we proceed as in case (c): At message 1 the H-buyer bids $d^1_{j_*} \cdot v_h$ instead of $\frac{1}{2}$ which increases his profit by $x^2 v_h$, and at message 2 he bids $d^2_{j_*-1} \cdot v_h$ instead of $\frac{1}{2} \cdot v_h$, keeping the same probability of winning and decreasing the winning price (for message 2) by $[\frac{1}{2} - d^2_{j_*-1}]v_h \ge [(\ell^1_{j_*} - x)]v_h$ (by the structure of $\sigma(m,n)$), thus increasing profit by at least $(\ell^1_{j_*} - x)y^2_{j_*-1}v_h > (\ell^1_{j_*} - x)\ell^2_1 v_h > (\delta_n - x)\delta_n v_h$ (by Lemma 10). Altogether the increase of profit is again greater than $[x^2 + (\delta_n - x)\delta_n]v_h$ whose minimum is $\frac{3}{4}\delta_n^2 v_h$.

This completes the proof of Theorem 9. □

References

ANDREONI, J., Y.-K. CHE, AND J. KIM (2007): "Asymmetric information about rivals' types in standard auctions: An experiment," *Games and Economic Behavior*, 59(2), 240–259.

ĀZACIS, H., AND P. VIDA (2015): "Collusive communication schemes in a first-price auction," *Economic Theory*, 58(1), 125–160.

BERGEMANN, D., B. BROOKS, AND S. MORRIS (2013): "Extremal Information Structures in the First Price Auction," *Princeton University William S. Dietrich II Economic Theory Center Research Paper*, (055-2013).

——— (2017): "First-Price Auctions With General Information Structures: Implications for Bidding and Revenue," *Econometrica*, 85(1), 107–143.

BERGEMANN, D., AND S. MORRIS (2019): "Information design: A unified perspective," *Journal of Economic Literature*, 57(1), 44–95.

BORDER, K. C. (1989): *Fixed Point Theorems with Applications to Economics and Game Theory*. Cambridge University Press.

FANG, H., AND S. MORRIS (2006): "Multidimensional private value auctions," *Journal of Economic Theory*, 126(1), 1–30.

HART, S., I. KREMER, AND M. PERRY (2017): "Evidence Games: Truth and Commitment," *American Economic Review*, 107(3), 690–713.

KAMENICA, E., AND M. GENTZKOW (2011): "Bayesian persuasion," *American Economic Review*, 101(6), 2590–2615.

KAPLAN, T., AND S. ZAMIR (2000): "The strategic use of seller information in private-value auctions," *Hebrew University, Center for the Study of Rationality, WP221*.

KAPLAN, T. R., AND S. ZAMIR (2015): "Multiple equilibria in asymmetric first-price auctions," *Economic Theory Bulletin*, 3(1), 65–77.

KIM, J., AND Y.-K. CHE (2004): "Asymmetric information about rivals' types in standard auctions," *Games and Economic Behavior*, 46(2), 383–397.

KIM, J., AND Y. KOH (2020): “Learning rivals’ information in interdependent value auctions,” *Journal of Economic Theory*, 187, 105029.

LANDSBERGER, M., J. RUBINSTEIN, E. WOLFSTETTER, AND S. ZAMIR (2001): “First-price auctions when the ranking of valuations is common knowledge,” *Review of Economic Design*, 6(3-4), 461–480.

LEBRUN, B. (2010): “First-price auctions with resale and with outcomes robust to bid disclosure,” *The RAND Journal of Economics*, 41(1), 165–178.

MARTÍNEZ-PARDINA, I. (2006): “First-price auctions where one of the bidders’ valuations is common knowledge,” *Review of Economic Design*, 10(1), 31–51.

MATHEVET, L., J. PEREGO, AND I. TANEVA (2020): “On information design in games,” *Journal of Political Economy*, 128(4), 1370–1404.

PERRY, M., E. WOLFSTETTER, AND S. ZAMIR (2000): “A Sealed-Bid Auction That Matches the English Auction,” *Games and Economic Behavior*, 33(2), 265–273.

SKRETA, V. (2015): “Optimal auction design under non-commitment,” *Journal of Economic Theory*, 159, 854–890, Symposium Issue on Dynamic Contracts and Mechanism Design.

SOROKIN, C., AND E. WINTER (2021): “Pure Information Design in Classic Auctions,” *Hebrew University Discussion paper-DP742*.

ZHENG, C. (2002): “Optimal auction with resale,” *Econometrica*, 70(6), 2197–2224.

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