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PURE INFORMATION DESIGN IN CLASSIC AUCTIONS

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Pure Information Design in Classic Auctions

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Abstract

In many auction environments sellers are better informed about bidders' valuations than the bidders themselves. For such environments we derive a sharp and general optimal policy of information transmission in the case of independent private values. Under this policy bidders whose (ex-post) valuation is below a certain threshold are provided with all the information (about their valuations), but those bidders whose valuation lies below the threshold receive no information whatsoever. Surprisingly, the threshold expressed in percentiles is independent of the probability distribution over bidders' ex-post valuations; it depends solely on the number of bidders. Similar results are also derived for the bidder-optimal policy. Our analysis builds on the approach of "Bayesian persuasion" and on a linearity of sellers' revenues as a function of the inverse distribution. This latter property allows us to use important results on stochastic comparisons.

Keywords. Information Design, Bayesian Persuasion, Mechanism Design, Auction, Optimal Allocation.

JEL Classification Numbers: C72, D44, D82.

1 Introduction

In many economic environments where auctions are used to sell a good or procure a service, the mechanism designer (henceforth "the seller") is better informed about bidders' valuations (or costs, in case of procurements) than the bidders themselves. One such prominent market is the market for online advertising space. Giant internet companies that sell ad space through auctions can typically predict the revenue that these ads generate even more accurately than the bidders do. Their ability to store and analyze data of past auctions and the habits of their platform users' is far superior to that of the companies that bid for ad space.

Outside the realm of online advertising the side that runs the auction often holds information regarding bidders' valuations that is superior in quality to the information

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held by the bidders themselves. A procurement that has been running repeatedly often allows the government to be better informed about the costs involved than bidding firms that participate for the first time. Trades of used goods that are put on the market through auctions (whether real estate, used cars, or commercial machines) typically involve a situation of asymmetric information. Sellers in such auctions possess better information about the quality of the good and its adequacy for different purposes than that of any of the potential buyers. Finally, big auction houses that sell art pieces, such as Sotheby's and Christy's, have their own team of experts and evaluators that advise their clients, and often know more about their lots than some of the participating bidders.

Yet in all the markets we mentioned above auctions remain the main trading mechanism in spite of the fact that they may defy the standard informational assumptions of auction models. In many environments auctions are used not necessarily for the purpose of revenue maximization but because they allow for transparency and make corruption[1] more difficult in other environments they are used as a substitute for the more costly mechanism of negotiations. We also point out that the mere use of reserve prices may be problematic for sellers as it implies a commitment by the seller not to sell the good under certain circumstances, which may appear impossible. In all these cases, whenever sellers possess more advanced information about buyers' valuations than the buyers themselves, they may be able to improve the revenue by committing to selectively transmit the available information to all or some of the buyers.

The purpose of this paper is to derive the optimal revelation policy of sellers in such situations. We address this issue here by mainly limiting ourselves to the case of independent private value auctions under the limit case where buyers know only the expected value of the object and the seller knows more.[2] We also assume that the seller is allowed to provide information about bidders' own valuations[3] but not about other bidders' valuations; the idea here is that the seller can provide hard evidence only regarding the object's attributes but not about someone else's subjective preferences, and so it is reasonable to assume that the verifiable information concerns only a bidder's own valuation. Furthermore, revealing private information about other bidders would be subject to legal constraints in many cases.

In addition to arguing that sellers often possess valuable information on bidders' ex-post valuations of the object sold, we stress that our results are also relevant in environments where sellers lack such information. Generally speaking, information design does not require information possession; however, it does require that the seller possess an instrument with which he can control the flow of information — and guarantee selective access to that information.

Quite surprisingly, in such environments we are able to provide a clear-cut, general, and accurate answer to the problem of the revenue-maximizing policy. It is clear-cut since,

[1] In some cases, such as government procurement, using mechanisms other than auctions is not an option due to counter-corruption regulation.

[2] Without loss of generality we assume that the seller knows the exact ex-post valuations of the buyers, as any symmetric lack of information is irrelevant.

[3] This is different from Kaplan and Zamir [5], who assume that the information regarding the ranking of players' valuations is made available by the seller.

as we shall see, the optimal policy prescribes sellers to either provide a bidder with all the information the seller has or none. It is general as it turns out that the optimal policy is totally independent of the probability distribution that generates bidders' ex-post valuations. Finally, it is accurate as it allows us to determine exactly which bidders will receive all the information, and which bidders will receive none. The criterion determining the two groups depends only on the number of bidders participating in the auction and not at all on the probability distribution generating bidders' ex-post valuations.

We find this precision and simplicity of the optimal policy to be rather exceptional in the auction theory literature. To be more formal, our result shows that for each number of bidders n, there is a probability $q^*(n)$ such that bidders whose ex-post valuation lies below the $q^*(n)$ percentile of the prior distribution receive all the information and the others receive none.

Furthermore, if the same policy is applied but the cutoff threshold q is different, then the seller's profits are increasing if $q < q^*(n)$ and decreasing otherwise. It's also always optimal for the seller to hide information from some small (theoretical) share of bidders with maximal valuations and to provide all the information to some small (theoretical) share of bidders with minimal valuations. Now we provide intuition for these claims.

Suppose that all the bidders are perfectly informed; we argue that it is possible for the seller to increase his revenue just by hiding information from the bidders with very high ex-post valuations. First, it is noteworthy that this distortion goes into effect[4] only when there are at least two bidders with such valuations. Second, the probability that there are three or more such bidders is significantly lower than the probability that there are exactly two. Finally, if there are exactly two such bidders, then it is the lowest valuation of the two that matters; by providing no information the seller decreases the gap between their valuations, thus increasing the auction revenue.

Similarly, if all the bidders have no information, then it is possible for the seller to increase his revenue by providing information to all of the bidders with very low ex-post valuations. Indeed, such revelation brings good news to the majority of the bidders — now they know that their valuations are not very low and are willing to bid more. Still there is a positive probability that even the second-highest valuation turns out to be very low, with negative effect on the auction revenue; however, with three or more bidders the probability of such event is too small to reverse the overall positive effect.

While our analysis does not rely on very advanced mathematical methods it does use a nonstandard approach that we believe might be useful in addressing other issues in auction theory. We express sellers' revenue not as a function of the probability distribution over valuations (as the standard auction theory literature does) but rather as an inverse function of the probability distribution. This allows us to make use of Blackwell's results that connect the degree of information possessed by bidders to the notion of mean-preserving spread, and utilize other important results from the literature on stochastic orders.

Our finding that the seller commits to inform only low types may seem somewhat counterintuitive, as it involves a seller's commitment to conceal good news and lose profits. However, such a commitment allows the seller to pool bidders with high valuations

[4] For clarity, we take the second-price auction as our main mechanism here.

together, while allowing those bidders, whose valuations are slightly above the threshold to be highly optimistic, thereby making the auction environment more competitive than the one that would prevail if all bidders were fully informed. As we shall see, while there are realizations under which the seller loses from this policy relative to the two extreme policies (where either all players are provided with full information or none of them receive any information), this policy increases the expected value of the second-order statistics and hence the seller's revenue.

Our paper also contributes to the more general literature on optimal object allocation initiated by Vickrey [12] and extended by Myerson [9]. Specifically, our paper studies the issue of information design as a separate task faced by the mechanism designer (i.e., on top of designing the rules of the game). The tern "information design" itself is coming from the pioneering work of Taneva [11].

Our paper is related to Bergemann and Pesendorfer [2]. Their paper considers a model in which the seller first designs the information structure and then the mechanism. This later stage is carried out without the seller having any access to signals. In our model we assume that sellers have no control over the mechanism or reserve prices[5] and that the optimization is carried out only with respect to the information structure. This allows us to establish much stronger results, providing a simple scheme that fully specifies the information each bidder receives, and proving that the optimal threshold does not depend on the distribution of the ex-post valuations.

Our approach is also closely related to the Bayesian persuasion literature and, in particular, to Kolotilin et al. [6]; like them our analysis uses the same approach to simplify the infinite-dimensional (optimal control) problem of finding the optimal posterior distribution and heavily relies on Blackwell's [3] characterization result of information structures. In their paper[6] the optimal information design is meant to affect the action of a single receiver, whereas in our case it is meant to affect the equilibrium of an auction game.

Our results apply to the case of independent bidder types and all efficient auction mechanisms (i.e., all mechanisms to which the revenue equivalence theorem applies); see Bergemann et al. [1] for the treatment[7] of the case with interdependent values.

2 The Model and the Results

Consider a classic auction model satisfying all the assumptions of the revenue equivalence theorem. A single object is being sold to one of $n > 1$ bidders. Let $\tilde{F}$ be a cumulative distribution function (or CDF for short) of a bidder's ex-post valuation $\tilde{\theta}$ with an expected value of E_θ and a well-defined nonatomic density[8] $\tilde{f}$ on the support $[0, \tilde{b}]$. The seller assigns zero value to the object. The distribution $\tilde{F}$ is common knowledge; however the bidders

[5] It prevents the seller from using a posted price at the highest valuation.

[6] One of the main objectives of their paper is to show the equivalence between optimal persuasion and experimentation.

[7] As Bergemann et al. [1] show, if the seller can control the interdependency of buyers' valuations, he can obtain almost full revenue extraction. They also provide lower bounds for the seller's revenue as a function of the joint distribution over valuations.

[8] From now on uppercase letters stand for CDFs, while lowercase ones stand for their densities.

do not know their valuations $\tilde{\theta}$ without additional information from the seller; they know only the ex-ante expected value, which is identical over all bidders. The bidders' utility is as usual linear in payment.

The seller can provide any information to the bidders about their valuations; however, he should treat them symmetrically, so that, if some bidder with an ex-post valuation of $\tilde{\theta}$ learns something, then any other bidder with the same valuation should get this information as well. The seller should also treat bidders independently, so that he can't condition the information he provides to bidder i on the valuation of some other bidder. Under these assumptions the resulting auction game is one of symmetric private value, so that the revenue equivalence theorem holds (up to a reserve price). Thus we will not be concerned here with the optimization of the mechanism itself; instead we regard it as fixed and without loss of generality we will assume it to be the second-price auction.

We model information structures in the following[9] manner. The seller has a set of signals (or messages) $S \in [0, \tilde{b}]$ that he can send to a bidder. These signals together with the set of bidders' types Θ form a measurable space $(\Theta \times S, \mathcal{B}(\Theta \times S))$, where $\mathcal{B}(\Theta \times S)$ is a class of Borel sets on $\Theta \times S$. An information structure is a pair $(S, \mathcal{F}(\cdot, \cdot))$, where S is the set of signals and $\mathcal{F}$ is a joint cumulative distribution on $\Theta \times S$. We refer to $\mathcal{F}$ as the seller's commitment: intuitively the seller commits to generate a random signal in such a way that $\mathcal{F}$ is the joint distribution over the pairs of signals and types. Of course, it must be the case that $\tilde{F}(x) = \mathcal{F}(x, +\infty)$.

Given the signal s_i and the information structure $(S, \mathcal{F}(\cdot, \cdot))$, each bidder forms an estimate about her true valuation of the object. The expected value of $\tilde{\theta}_i$ conditional on observing s_i is given by

$$\hat{\theta}(s_i) = \mathbb{E}[\tilde{\theta}_i | s_i] = \int_0^1 y d\mathcal{F}(y|s_i).$$

Every information structure S generates a distribution function $F(\theta)$ over posterior expectations θ given by

$$F(x) = \int_{s:\hat{\theta}(y) \leq x} d\hat{F}(y),$$

where $\hat{F}(\cdot)$ is the marginal of $\mathcal{F}(\cdot, \cdot)$ with respect to s. Once the seller publicly commits to the information revelation procedure (or a persuasion mechanism), an auction is being conducted with each bidder valuation being the conditional expected value (derived by Bayesian updating).

We can now restate the main objective of our paper in a more formal manner: find the seller's optimal information transmission policy (or persuasion procedure). The optimization problem stated here is one where the unknown variable is a function (rather than a real number). Formally, it represents a rather complex optimal-control problem. However, we were able to overcome most of the technical hurdles with the help of an elegant geometric argument, one that may prove useful in future research on optimization over information structures.

[9] We follow Bergemann and Pesendorfer [2] closely here.

Theorem 1 *The seller-optimal information transmission policy in the case of* $n \geq 3$ *has the following structure: there exists a probability* $q^* \in [0,1]$ *such that:*

- *Low-types bidders (with ex-post valuations below* $\tilde{F}^{-1}(q^*)$*) learn their exact valuations.*
- *High-type bidders (with ex-post valuations above* $\tilde{F}^{-1}(q^*)$*) receive no information at all (and hence only learn that their valuations are above* $\tilde{F}^{-1}(q^*)$*).*
- *This cutoff probability* q^* *can be derived from the following equation:*

$$(n-1)^2 q^n - (2n-3)nq^{n-1} + n(n-1)q^{n-2} = 1.$$

- *The solution* q^* *is increasing in* n*; when* n *goes to infinity,* q^* *goes to 1; i.e. everyone receives all the information.*
- *If the same structure of information transmission is used with any other threshold* q*, then the seller's revenue is increasing in* q *for* $q \in [0, q^*]$ *and decreasing for* $q \in [q^*, 1]$*.*

We provide solutions for this equation for all $n \leq 10$ in Table 1 below.

Remark 1 *For* $n = 2$ *we have* $q^* = 0$*; i.e. no information is provided at all.*

Two remarkable facts are implied by Theorem 1. First, the seller-optimal information transmission policy is simple in that the seller either transmits all the information or none. Second, the threshold probability is independent of the distribution. Taken together, these two properties allow us to make clear-cut policy recommendations for profit-maximizing sellers: provide all the information to the low types and hide information from the high types. We now provide a sketch of the proof whose the key idea is presented in Figure 1.

We start with the standard textbook expression for the expected revenue of the seller, where the posterior CDF F of players' valuations plays a prominent role. Due to an elegant result by Blackwell [3], finding the optimal information structure boils down to finding the CDF that maximizes the revenue expression among a class of CDFs all of which have the same mean and can be ranked by the order of mean-preserving contraction. However, solving for the optimal F directly turns out to be intractable. A major trick in our analysis is to express the revenue in terms of the inverse CDF. Using integration by substitution and integration by parts we express the seller's revenue as a function of F^{-1} (i.e. the inverse of the CDF F) which allows us to proceed. As it turns out this expression for the ex-ante expected payment (EP) has the following form

$$EP = \int_0^1 g(x)v(x)dx$$

where $v(x) = \int_0^x F^{-1}(y)dy$ and $g(x)$ is a function that has a single root x^* in $[0,1]$ and is positive above x^* and negative below it. Our task is now to find the function $v(x)$ that will maximize the expression EP. But because of the multiplicative structure of the integrand of EP it is clear that $v(x)$ has to be as high as possible as we get closer to 1 and as low as possible as we get closer to zero. Furthermore, in order for $v(x)$ to correspond to (the integral of an inverse function of) some CDF it needs to satisfy some additional restrictions (in particular, it needs to be increasing and convex).

A well-known result in probability theory [10] asserts that for every two CDFs F and $\tilde{F}$, $\tilde{F}$ is a mean-preserving spread of F if and only if $v(x) \geq \tilde{v}(x)$ for every x. As Figure 1 illustrates, the proof shows that:

- The function $\tilde{v}(x)$ corresponds to the CDF of full information (green),
- The function $\hat{v}(x)$ corresponds to the CDF of zero information (a CDF where the mean value occurs with probability 1) (blue),
- The function $g(x)$ (teal),
- The optimal function $v(x)$ must convex, increasing, and fall between the blue line and the green curve.

Without convexity constraints on $v(x)$ it is clear that the function would coincide with the blue line for values above x^* (where g is positive) and would coincide with the green curve on values below x^* (where g is negative). Unfortunately, this function is not convex and hence cannot be derived from a CDF. Still, any convex function $v(x)$ maximizing EP must be linear on the interval on the right side of x^*. If this is not the case an improvement can be demonstrated by invoking a linear function in this region. Because g is negative on the left side of x^* there will be some optimal slope for this linear segment of $v(x)$. Depending on this slope define q^* to be the point where the linear segment (seen in red) intersects the green curve. Clearly, the left side of q^* $v(x)$ must coincide with the green curve to minimize the effect of the negative value of g on the value of the integral. Hence, it is left to determine q^*. The value of q^* determines the threshold we mentioned in the Introduction; i.e., all types above $\tilde{F}^{-1}(q^*)$ are uninformed and all types below $\tilde{F}^{-1}(q^*)$ are fully informed.

Given the properties of $v(x)$ we have already obtained, we are able to express EP as a function of q^*, and write down the first-order conditions determining its value. Our surprising finding is that these conditions boil down to finding a unique root of a polynomial in n on $[0,1]$. This implies that the threshold q^* depends only on the number of bidders and not on the initial CDF.

The same proof technique described above can also provide us with the dual result of Theorem 1: suppose that the buyers can organize and introduce a credible commitment mechanism that ignores information under certain circumstances (for example, by appointing a mediator who receives the information, screens it, and then retransmits it according to the agreed policy). Then we can derive the optimal buyers' policy, which is even simpler than the seller's optimal policy. This is done in Theorem 2.

Theorem 2 *The bidder-optimal information transmission policy has the following properties:*

- *Low-type bidders (with an ex-post valuation below $\tilde{F}^{-1}(q^{\#})$) receive no information from the mediator save for the fact that their valuation is below the threshold.*

- *High-type bidders (with an ex-post valuation above $\tilde{F}^{-1}(q^{\#})$) receive all the information from the mediator.*
- *The threshold probability $q^{\#}$ equals $\frac{n-2}{n-1}$.*
- *If the same structure of information transmission is used with any other threshold q, then the buyers' utility is increasing in q for $q \in [0, q^{\#}]$ and decreasing for $q \in [q^{\#}, 1]$.*

The information distortion clearly decreases social welfare, where one side can benefit only by inducing additional costs for the other; it is impossible to increase both the seller's expected profit and the buyer's ex-ante utility by concealing information.

3 Pre-informed bidders

In this section we consider a situation when some information cannot be hidden from the bidders. The seller is still aware of all the information about bidders' valuations and can provide it to the bidders in any feasible manner, but now the bidders receive additional signal about their valuations. The seller is aware[10] of the value of this signal too, but cannot hamper it in any way. This case stands for the situation when some information has to be revealed to the bidders.

To accommodate this assumption our main model is amended the following way. Prior to receiving a signal s_i coming from the seller, each bidder receives their own signal of $\gamma \in \Gamma$. Let $f_{v,\gamma}(x,y)$ be a joint density of the ex-post valuation of v_i and signal γ_i, - it should be the same for all players due to symmetry assumption. The signal coming from the seller can depend not only on v, but on γ too.

Let $(V \times \Gamma \times S, \mathcal{B}(V \times \Gamma \times S))$ be a measurable space, where $\mathcal{B}(V \times \Gamma \times S)$ is the class of Borel sets of $V \times \Gamma \times S$. A persuasion mechanism (or information structure) is a pair $(S, F_{v,\gamma,s}(\cdot))$, where S is the space of signal realizations, and $F_{v,\gamma,s}$ is a joint probability distribution over the space of valuations V, space of private signals Γ, and the space of signals S. Of course, to maintain Bayesian plausibility we should have $f_{v,\gamma}(x,y) = \int f_{v,\gamma,s}(x,y,z)dz$. Equivalently, we can define persuasion mechanism using conditional distribution of s_i given v_i and γ_i.

Given the signal s_i, his private information γ_i, and the information structure each bidder forms an estimate about her true valuation of the object. The expected value of v_i conditional on observing s_i and γ_i is given by

$$\theta_i(\gamma_i, s_i) = \mathbb{E}[v_i|\gamma_i, s_i].$$

Every information structure generates a distribution function $F_{\theta_i}(\cdot)$ over posterior expectations

$$F_{\theta_i}(x) = \int_{y,z:\theta_i(y,z)\leq x} dF_{\gamma_i,s}(y,z).$$

[10] If the seller is not aware of this signals, then Blackwell's [3] characterization of posterior distributions no longer applies and the analyses becomes much more complicated.

It's useful to consider two limit cases: Posterior for full information: $\tilde{F}_{\theta_i}(x) = F_v(x)$. and posterior for no information: $\hat{F}_{\theta_i}(x) = \int_{z:\theta_i(z)\leq x} dF_{\gamma_i}(z)$. ($\theta_i(\gamma_i, s_i)$ does not depend on s_i). As in we did in the previous section, let us define $\hat{v}(x) = \int_0^x \hat{F}_{\theta_i}^{-1}(z)dz$, and $\tilde{v}(x) = \int_0^x \tilde{F}_{\theta_i}^{-1}(z)dz$.

Theorem 3 *The seller-optimal persuasion posterior in case of $n > 2$ has the following structure:*

- *If $\hat{F}^{-1}(1) \geq \frac{\hat{v}(1)-\tilde{v}(q^*)}{1-q^*}$, then the posterior obtained in theorem 1 is feasible and optimal.*
- *Otherwise, there exist $0 < q^* < p^* < 1$ such that the integral of the inverse of optimal posterior has the following structure:*

$$v(x) = \begin{cases} \tilde{v}(x), & x \leq q, \\ \frac{\hat{v}(p)-\tilde{v}(q)}{p-q}(x-q) + \tilde{v}(q), & p > x > q, \\ \hat{v}(x), & x \geq p. \end{cases}$$

- *The values of q^* and p^* can be found from the following system:*

$$\begin{cases} \int_q^p (x - \frac{n-2}{n-1})x^{n-3}\,(p-x)\,dx = 0, \\ \hat{v}'(p)(p-q) = \hat{v}(p) - \tilde{v}(q). \end{cases}$$

- *Note that the first equation is a polynomial, just as before, while the second one does depend on the model primitives – (integrals of inverse) distributions of types with all and no information.*

4 Discussion

A central assumption in our paper is that buyers do not possess private information prior to the revelation by the seller. If such information is available to buyers then the optimal information design by the seller is much harder to derive. Unfortunately, our techniques are insufficient to solve this problem in the general case. In particular, Blackwell's [3] result does not apply anymore. However, we can show that if buyers' private information is sufficiently inferior to that of the seller (especially as far as high types are concerned), then our results hold.

Another question worthy of addressing concerns the combined design of information and reserve prices. As we noted in the Introduction, our model assumes that the auction mechanism is fixed and cannot be controlled by the seller. One might wonder nevertheless what the optimal reserve price might be after the seller has revealed the information (according to the optimal policy proposed here). Unfortunately, Myerson's [9] optimal auction paper doesn't give an answer to this question as the resulting (ex-post) distribution following the revelation does not satisfy the smoothness condition. However, Bergemann and Pesendorfer [2] turns out to be more helpful here and implies that the reserve price

Number of bidders	3	4	5	6	7	8	9	10
Seller-optimal q^*	0.25	0.46	0.58	0.66	0.72	0.75	0.78	0.81
Optimal revenue	0.55	0.63	0.69	0.73	0.76	0.79	0.81	0.83
Revenue under complete information	0.50	0.60	0.67	0.71	0.75	0.78	0.80	0.82
Information effect on revenue, %	10.5	4.78	2.91	2.03	1.54	1.23	1.02	0.87
Reserve price effect on revenue, %	6.25	2.08	0.78	0.31	0.13	0.05	0.02	0.01
Ratio of effects	1.68	2.29	3.72	6.51	11.8	22.1	42.1	80.8
Buyer-optimal threshold $q^{\#}$	0.5	0.67	0.75	0.8	0.83	0.86	0.88	0.89
Buyer-optimal utility	0.09	0.06	0.05	0.04	0.03	0.03	0.02	0.02
Buyers' utility under complete information	0.08	0.05	0.03	0.02	0.02	0.01	0.01	0.01
Effect of information on buyers, %	6.25	19.7	35.6	52.4	69.8	87.4	105	123

Table 1: Effects of information design (ID)

should be set at the highest value ex post, which is the expected value conditional on being above the threshold. It is important to note, however, that once such a reserve price is set the information design ceases to be optimal (as our model assumes the absence of reserve prices).

Regardless of the structure of the seller's combined policy it would be instructive to assess the effectiveness of optimizing over the information structure alone as opposed to optimizing over the reserve price alone. Table 1 provides the relevant numbers for the case of uniform distribution for different auction sizes (note that while the thresholds are independent of the distribution over valuations the optimal revenue does depend on the distribution). We observe that in this case controlling the information structure has a much greater effect on revenues then determining the reserve prices.

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Appendix

Proof of Theorem 1.

First we represent the expected payment and the expected bidders' utility as a linear function of the inverse[11] $F^{-1}(q)$ of the posterior distribution $F(\theta)$. To do so, consider a standard n-bidder auction that satisfies all the assumptions of the revenue-equivalence theorem. Let $F(x)$ be the CDF of the bidder's type. In this case an expected payment of a type x bidder equals:

$$\int_0^x yg(y)dy,$$

where $g(y) = (n-1)F^{n-2}(y)f(y)$.

To obtain the ex-ante expected payment of the bidder, we need to integrate this value into the distribution of the bidder's type:

$$\int_0^1 \int_0^x y(n-1)F^{n-2}(y)f(y)dyf(x)dx.$$

By changing the order of integration and substituting $x = F^{-1}(y)$ we obtain the following formula for the bidder's ex-ante expected payment:

$$EP = (n-1)\int_0^1 (1-y)y^{n-2}F^{-1}(y)dy.$$

[11] Standard textbooks, like Krishna (2010) provide the original expressions; the only thing that remains is to change variables.

A similar expression can be obtained for the bidder's ex-ante expected utility:

$$EU = \int_0^1 \left[x^{n-1} - (n-1)(1-x)x^{n-2}\right] F^{-1}(x)dx.$$

We now turn to the posterior distribution $F(\cdot)$ of the bidders' conditional expected values. This distribution has one prominent property: it is a mean-preserving contraction (MPC) of the distribution $\tilde{F}$ of ex-post valuations (see Blackwell [3]). The converse also holds true: every MPC of $\tilde{F}$ can be obtained by some information transmission policy $(S, \mathcal{F}(\tilde{\theta}, s))$.

We now use the following characterization of MPC:

Statement 1 *(Shaked and Shanthikumar, 2007, p. 112)] Let θ and $\tilde{\theta}$ be two random variables with CDFs F and $\tilde{F}$. Suppose that $F^{-1}(\cdot)$ and $\tilde{F}^{-1}(\cdot)$ are well defined. Then*

$$\begin{cases} \int_0^q F^{-1}(y)dy \geq \int_0^q \tilde{F}^{-1}(y)dy, & \forall q \in [0,1), \\ \int_0^1 F^{-1}(y)dy = \int_0^1 \tilde{F}^{-1}(y)dy. \end{cases}$$

if and only if θ is a mean-preserving contraction of $\tilde{\theta}$.

We now define the functions $v(x)$ and $\tilde{v}(x)$ as follows:

$$v(x) = \int_0^x F^{-1}(y)dy, \qquad \tilde{v}(x) = \int_0^x \tilde{F}^{-1}(y)dy.$$

In what follows we shall optimize the seller's revenue with respect to v instead of F^{-1}. Integrating by parts we get

$$EP = (n-1)^2 \int_0^1 (x - \frac{n-2}{n-1})x^{n-3}v(x)dx, \tag{1}$$

$$EU = E_F - n(n-1)\int_0^1 (x - \frac{n-2}{n})x^{n-3}v(x)dx.$$

The following observation trivially follows from the above expressions:

Observation 1 *The coefficient of $v(x)$ in equation 1 is positive for high values of x and negative for low ones. Hence, for large enough x, $v(x)$ has to be as large as possible (but not above the line connecting $(1, E_F)$ and $(\frac{n-2}{n-1}, \tilde{v}(\frac{n-2}{n-1}))$. Hence , in that area, $v(x)$ has to be linear. Hence, the optimal policy prescribes zero information for sufficiently high types.*

Now recall that F is a result of an information design procedure and, therefore, a mean-preserving contraction of a true distribution $\tilde{F}$. Hence, v has to satisfy the following constraints:

$$v(x) \geq \tilde{v}(x), \quad v(0) = \tilde{v}(0), \quad v(1) = \tilde{v}(1), \quad \tilde{v}(x) = \int_0^x \tilde{F}^{-1}(y)dy.$$

As the most "contracted" distribution is just a constant (no information at all is provided), we have that

$$v(x) \leq xE_F, \qquad E_F = E(\tilde{\theta}) = v(1) = \int_0^1 F^{-1}(y)dy.$$

We now have the following optimization problem:

$$\begin{cases} (n-1)^2 \int\limits_0^1 (x - \frac{n-2}{n-1}) x^{n-3} v(x) dx & \to \max\limits_{v(x)}, \\ \text{s.t.} \\ \tilde{v}(x) \leq v(x) \leq xE_F, \\ v(0) = 0, \ v(1) = E_F, \\ v(x) - \text{non-decreasing}, \\ v(x) - \text{convex}. \end{cases}$$

Using the argument in Observation 1, we note that optimal $v(x)$ has to be a straight line on the interval $[\frac{n-2}{n-1}, 1]$; otherwise, we can make it so and thereby increase the integral value in a region where the intergand is positive and thereby increase the revenue. Next, on the left side of $\frac{n-2}{n-1}$, we want $v(x)$ to be as low as possible (but above $\tilde{v}(x)$). (Ideally we would have liked it to coincide with $\tilde{v}(x)$; however, this violates the convexity constraint.) Therefore, $v(x)$ has to be linear with the slope of $v'(\frac{n-2}{n-1})$ (right derivative) until it hits $\tilde{v}(x)$. But the highest possible value of $v'(\frac{n-2}{n-1})$ is

$$\frac{E_F - v(\frac{n-2}{n-1})}{1 - \frac{n-2}{n-1}}$$

due to the convexity constraint. We therefore conclude that the optimal $v(x)$ satisfying all the constraints is given by

$$v(x) = \begin{cases} \tilde{v}(x), & x \leq q, \\ \frac{E_F - \tilde{v}(q)}{1-q}(x-q) + \tilde{v}(q), & x > q. \end{cases}$$

Our optimization problem now boils down to finding the optimal cutting point where v turns from a linear to a nonlinear function. To solve this we need to maximize the following expression with respect to q:

$$EP(q) = (n-1)^2 \int_0^q (x - \frac{n-2}{n-1}) x^{n-3} \tilde{v}(x) dx + \\ + (n-1)^2 \int_q^1 (x - \frac{n-2}{n-1}) x^{n-3} \left[\frac{E_F - \tilde{v}(q)}{1-q}(x-q) + \tilde{v}(q) \right] dx. \quad (2)$$

We now need to solve for the first-order condition and hence to take derivative. Note that the integrand is continuous at $x = q$ for every q; thus, differentiating the above expression is quite simple: the terms standing for differentiation of the upper and lower integration limits cancel each other out and the only thing that remains is direct differentiation of the integrand:

$$\frac{d}{dq}EP(q) = (n-1)^2(q - \frac{n-2}{n-1})q^{n-3}\tilde{v}(q) -$$

$$- (n-1)^2(q - \frac{n-2}{n-1})q^{n-3}\left[\frac{E_F - \tilde{v}(q)}{1-q}(q-q) + \tilde{v}(q)\right] +$$

$$+ (n-1)^2 \int_q^1 (x - \frac{n-2}{n-1})x^{n-3}\frac{d}{dq}\left[\frac{E_F - \tilde{v}(q)}{1-q}(x-q) + \tilde{v}(q)\right] dx. \quad (3)$$

Let's tackle the key expression separately:

$$\frac{d}{dq}\left[\frac{E_F - \tilde{v}(q)}{1-q}(x-q) + \tilde{v}(q)\right] = \frac{(x-1)\left((q-1)\frac{d}{dq}\tilde{v}(q) + E_F - \tilde{v}(q)\right)}{(1-q)^2}.$$

We obtain

$$\frac{d}{dq}EP(q) =$$

$$= \frac{(n-1)^2\left(E_F - \tilde{v}(q) - (1-q)\frac{d}{dq}\tilde{v}(q)\right)}{(1-q)^2}\int_q^1 (x - \frac{n-2}{n-1})x^{n-3}(x-1)\,dx. \quad (4)$$

A trivial geometric exercise shows that for nondegenerate $\tilde{v}(x)$ and $q \in (0,1)$, the following inequality holds;

$$\frac{(n-1)^2\left(E_F - \tilde{v}(q) - (1-q)\frac{d}{dq}\tilde{v}(q)\right)}{(1-q)^2} > 0.$$

It is true due to the fact that for any two points on a convex function the higher point must lie in the upper subspace of the tangential line of the other point (recall that $\tilde{v}$ is increasing and convex with $\tilde{v}(1) = E_F$).

Hence the sign of $\frac{d}{dq}EP(q)$ is determined by the expression:

$$K(q) = \int_q^1 (x - \frac{n-2}{n-1})x^{n-3}(x-1)\,dx.$$

Note that $K(1) = 0$, $K(0) = \frac{1}{n(n-1)^2}$, and

$$\begin{cases} (x - \frac{n-2}{n-1})x^{n-3}(x-1) < 0, & x > \frac{n-2}{n-1}, \\ (x - \frac{n-2}{n-1})x^{n-3}(x-1) > 0, & x < \frac{n-2}{n-1}. \end{cases}$$

Therefore the equation $K(q) = 0$ has exactly one root q^* on the $[0,1]$ interval and it stands for the maximum of $EP(q)$. Note that we also obtain that $EP(q)$ is increasing if $q < q^*$ and decreasing if $q > q^*$.

Note also that as $\frac{n-2}{n-1}$ is increasing in n, so is the root $q^*(n)$ of $K(q)$.

Finally, by direct integration we obtain

$$K(q) = 0 \Leftrightarrow (n-1)^2 q^n - (2n-3)nq^{n-1} + n(n-1)q^{n-2} = 1.$$

The proof of **Theorem 2** follows a somewhat similar exercise. Consider the expected utility function:

$$EU = E_F - n(n-1)\int_0^1 (x - \frac{n-2}{n})x^{n-3}v(x)dx.$$

Once again, we can derive that the optimal v function has the following structure:

$$v(x) = \begin{cases} \frac{\tilde{v}(q)}{q}x, & x < q, \\ \tilde{v}(x), & x \geq q. \end{cases}$$

By substituting it into the EU function and taking the derivative with respect to q we obtain:

$$\begin{aligned} &\frac{d}{dq}Eu(q) = \\ &= -n(n-1)\frac{d}{dq}\left[\int_0^q (x - \frac{n-2}{n})x^{n-3}\frac{\tilde{v}(q)}{q}xdx + \int_q^1 (x - \frac{n-2}{n})x^{n-3}\tilde{v}(x)dx\right] = \\ &= -n(n-1)\int_0^q (x - \frac{n-2}{n})x^{n-3}\frac{q\frac{d}{dq}\tilde{v}(q) - \tilde{v}(q)}{q^2}xdx = \\ &= -n(n-1)\frac{q\frac{d}{dq}\tilde{v}(q) - \tilde{v}(q)}{q^2}\int_0^q (x - \frac{n-2}{n})x^{n-2}dx = \\ &= -\left(q\frac{d}{dq}\tilde{v}(q) - \tilde{v}(q)\right)q^{n-3}\left(nq - n - q + 2\right) = \\ &= \left(q\frac{d}{dq}\tilde{v}(q) - \tilde{v}(q)\right)q^{n-3}(n-1)\left(\frac{n-2}{n-1} - q\right). \end{aligned} \quad (5)$$

The final expression of the derivation gives us the theorem's result: as $q\frac{d}{dq}\tilde{v}(q) - \tilde{v}(q)$ is always no less than zero, we have that $q^{\#} = \frac{n-2}{n-1}$, that it is a maximum point, and that the expected utility function is monotonically increasing to the left of it and decreasing to the right.

Proof of Theorem 3.

As compared to the proof of Theorem 1, this time we have one more constraint — if the seller chooses mechanism that reveals no information, then the buyers still have some private signal that allows non-trivial distribution of their conditional valuations. Similarly to the [6] paper, this results in the additional constraint on the posterior we are able to obtain as a result in the information disclosure. Exactly, the posterior has to be a mean-preserving contraction of the distribution of ex-post values. However, it has to be a mean-preserving spread of the distribution of ex-ante values, as the seller cannot hide all the information and squeeze the distribution into just one point.

Repeating the same steps as in the proof of Theorem 1, and using Statement 1, we arrive at the following optimization problem:

Now we have to consider the following problem:

$$\begin{cases} (n-1)^2 \int\limits_0^1 (x - \frac{n-2}{n-1}) x^{n-3} v(x) dx & \to \max\limits_{v(x)}, \\ \tilde{v}(x) \le v(x) \le \hat{v}(x), \\ v(0) = 0,\ v(1) = E_F, \\ v(x) - \text{increasing}, \\ v(x) - \text{convex}. \end{cases}$$

Now there are two possible situations. Either the constraint $v(x) \le \hat{v}(x)$ is non-binding, i.e., due to convexity of all the "v" functions, $\hat{F}^{-1}(1) \ge \frac{\hat{v}(1)-\tilde{v}(q^*)}{1-q^*}$ (q^* here is the opti mal probability threshold from the proof of Theorem 1). In the other case, by argument similar to the one in the proof of Theorem 1 we obtain the existence of threshold probability

the following functional form for the integral of posterior we are after:

$$v(x) = \begin{cases} \tilde{v}(x), & x \le q, \\ \frac{\hat{v}(p)-\tilde{v}(q)}{p-q}(x-q) + \tilde{v}(q), & p > x > q, \\ \hat{v}(x), & x > p. \end{cases}$$

Also note that:

$$\hat{v}'(p) = \frac{\hat{v}(p) - \tilde{v}(q)}{p-q}.$$

So that we have

$$\begin{aligned} EP(q,p) = (n-1)^2 \int_0^q (x - \frac{n-2}{n-1}) x^{n-3} \tilde{v}(x) dx + \\ + (n-1)^2 \int_q^p (x - \frac{n-2}{n-1}) x^{n-3} \left[\frac{\hat{v}(p) - \tilde{v}(q)}{p-q}(x-q) + \tilde{v}(q) \right] dx + \\ + (n-1)^2 \int_p^1 (x - \frac{n-2}{n-1}) x^{n-3} \hat{v}(x) dx. \quad (6) \end{aligned}$$

We have to optimize this expression with respect to p and q. We begin with calculating partial derivatives, first in q:

$$\begin{aligned} \frac{\partial}{\partial q} EP(q,p) = (n-1)^2 \int_q^p (x - \frac{n-2}{n-1}) x^{n-3} \frac{\partial}{\partial q} \left[\frac{\hat{v}(p) - \tilde{v}(q)}{p-q}(x-q) + \tilde{v}(q) \right] dx = \\ = \frac{(n-1)^2 \left((p-q)\, \tilde{v}'(q) - \hat{v}(p) + \tilde{v}(q) \right)}{(p-q)^2} \int_q^p (x - \frac{n-2}{n-1}) x^{n-3} (p-x)\, dx. \quad (7) \end{aligned}$$

And then in p:

$$\begin{aligned} \frac{\partial}{\partial p} EP(q,p) = (n-1)^2 \int_q^p (x - \frac{n-2}{n-1}) x^{n-3} \frac{\partial}{\partial p} \left[\frac{\hat{v}(p) - \tilde{v}(q)}{p-q}(x-q) + \tilde{v}(q) \right] dx = \\ = \frac{(n-1)^2 \left((p-q)\, \hat{v}'(p) - \hat{v}(p) + \tilde{v}(q) \right)}{(p-q)^2} \int_q^p (x - \frac{n-2}{n-1}) x^{n-3} (x-q)\, dx. \quad (8) \end{aligned}$$

Now let's differentiate the constraint:

$$G(q,p) = \hat{v}'(p)(p-q) - (\hat{v}(p) - \tilde{v}(q)) = 0.$$

$$\frac{\partial}{\partial q} G(q,p) = \tilde{v}'(q) - \hat{v}'(p)$$

$$\frac{\partial}{\partial p} G(q,p) = \hat{v}''(p)(p-q)$$

Thus we obtain the following system:

$$\begin{cases} \frac{(n-1)^2((p-q)\tilde{v}'(q)-\hat{v}(p)+\tilde{v}(q))}{(p-q)^2} \int_q^p (x - \frac{n-2}{n-1})x^{n-3}\,(p-x) = \lambda(\tilde{v}'(q) - \hat{v}'(p)), \\ \frac{(n-1)^2((p-q)\hat{v}'(p)-\hat{v}(p)+\tilde{v}(q))}{(p-q)^2} \int_q^p (x - \frac{n-2}{n-1})x^{n-3}\,(x-q)\,dx = \lambda\hat{v}''(p)(p-q) \\ \hat{v}'(p)(p-q) - (\hat{v}(p) - \tilde{v}(q)) = 0. \end{cases}$$

This system boils down to the following:

$$\begin{cases} \int\limits_q^p (x - \frac{n-2}{n-1})x^{n-3}\,(p-x)\,dx = 0, \\ \hat{v}'(p)(p-q) = \hat{v}(p) - \tilde{v}(q). \end{cases}$$

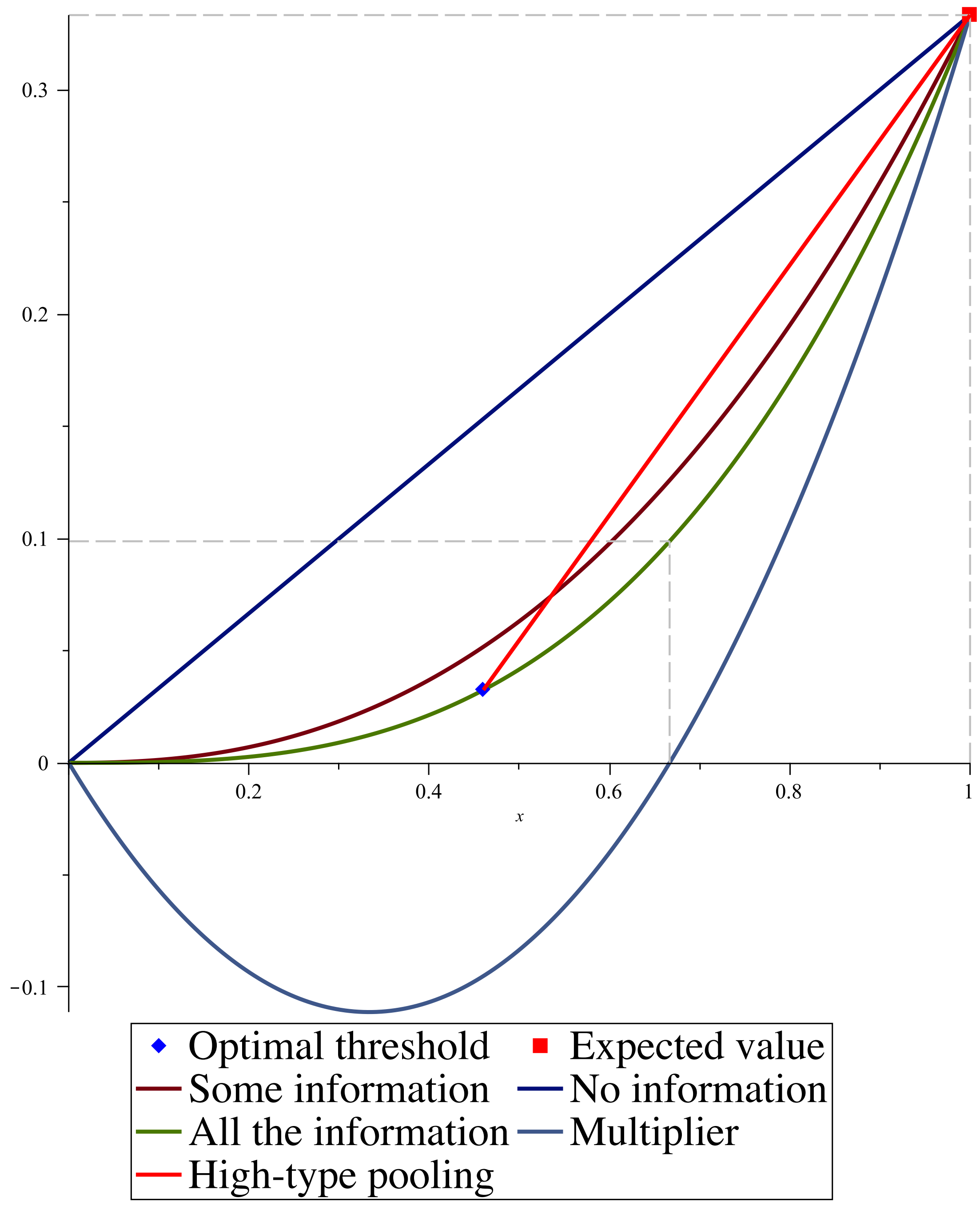

Figure 1: Formula (1) when $n = 4$ and $F^{-1}(q) = q^2$.

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