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**THE EFFECT OF SAMPLE SIZE AND
COGNITIVE STRATEGY ON PROBABILITY
ESTIMATION BIAS**

By

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The Effect of Sample Size and Cognitive Strategy on Probability Estimation Bias

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Abstract

Probability estimation is an essential cognitive function in perception, motor control, and decision making. Many studies have shown that when making decisions in a stochastic operant conditioning task, people and animals behave as if they underestimate the probability of rare events. It is commonly assumed that this behavior is a natural consequence of estimating a probability from a small sample, also known as sampling bias. The objective of this paper is to challenge this common lore. We show that in fact, probabilities estimated from a small sample can lead to behaviors that will be interpreted as underestimating or as overestimating the probability of rare events, depending on the cognitive strategy used. Moreover, this sampling bias hypothesis makes an implausible prediction that minute differences in the values of the sample size or the underlying probability will determine whether rare events will be underweighted or overweighted. We discuss the implications of this sensitivity for the design and interpretation of experiments. Finally, we propose an alternative sequential learning model with a resetting of initial conditions for probability estimation and show that this model predicts the experimentally-observed robust underweighting of rare events.

Keywords: Probability Estimation, Underweighting of Rare Events, Decision Making, Reinforcement Learning

The Effect of Cognitive Strategy and Sample Size on Probability Estimation

Probability Estimation

Explicit or implicit estimation of probabilities is involved in various cognitive tasks, including perception (e.g., Ashourian & Loewenstein, 2011; Knill & Richards, 1996; Raviv, Ahissar, & Loewenstein, 2012), sensory-motor integration (e.g., Körding & Wolpert, 2004), operant learning and decision making (e.g., Fox & Tversky, 1998; Kahneman & Tversky, 1979; Körding, 2007; Shafir, Reich, Tsur, Erev, & Lotem, 2008), and everyday cognition (e.g., Griffiths & Tenenbaum, 2006). However, the cognitive strategies underlying probability estimation from experience are poorly understood even in the simplest form of probability estimation: the estimation of the probability underlying a stationary sequence of binary trials, also known as a stationary Bernoulli process (e.g., Zhang & Maloney, 2012). The latter type of estimation is the focus of this paper.

One explicit experimental approach to the study of how people estimate the Bernoulli parameter from a finite sequence of Bernoulli trials is to present participants with different Bernoulli sequences and ask them to report the underlying probability (Erlick, 1964). Over the years there have been conflicting reports about biases in probability estimation in this task (Peterson & Beach, 1967): there are reports of underestimation of small probabilities (Pitz, 1965), overestimation of these probabilities (Erlick, 1964) and unbiased estimates (Ungemach, Chater, & Stewart, 2009). A caveat of this approach is that the task of explicitly reporting probabilities may be unnatural to the participants (Kyburg, 1996).

Probability estimation can also be examined indirectly, in the context of decision making (but see (Barron & Ursino, 2013; Barron & Yechiam, 2009; A. R. Camilleri & Newell, 2009; Friedman & Massaro, 1998; Gonzalez & Wu, 1999)). In particular, probability estimation is commonly studied using the *safe-risky* paradigm, in which participants repeatedly choose between two, unlabeled, alternatives. Undisclosed to the

participants, one choice, denoted as safe, returns a deterministic reward while the other, denoted as risky, returns a stochastic reward, such that the two actions return equal expected rewards (Barron & Erev, 2003; Barron & Ursino, 2013; Erev & Haruvy, 2009; Erev et al., 2010; Erev, Glozman, & Hertwig, 2008; Hertwig, Barron, Weber, & Erev, 2004; Shafir et al., 2008). The risky alternative is commonly chosen to be a Bernoulli variable with a probability p of a reward larger than the reward associated with the safe option and a probability $1-p$ of a reward smaller than the safe one.

Choice preference is usually quantified by computing the fraction of trials in which the risky alternative is chosen and/or the fraction of participants who choose the risky alternative more often (Shafir et al., 2008). Because the expected return associated with the safe and risky alternatives is equal, preference of the risky alternative by a risk-neutral participant is interpreted as an overestimation of p whereas preference of the safe alternative is interpreted as an underestimation of the Bernoulli parameter (Barron & Erev, 2003; March, 1996). To control for risk-averse or risk-seeking utility functions, choice preference in blocks of trials that differ in their value of p are compared (Hertwig, 2012).

Underweighting of Rare Events

There is substantial experimental evidence that in the safe-risky paradigm, preference for the risky alternative increases with the probability of the high risky outcome p (despite the equal returns). In other words, participants behave as though they underestimate the probability p when it is small and overestimate it when it is high, an effect that was termed the “*underweighting of rare events*” (Barron & Erev, 2003; Barron & Ursino, 2013; Barron & Yechiam, 2009; A. R. Camilleri & Newell, 2009; Erev et al., 2010; Hau, Pleskac, Kiefer, & Hertwig, 2008; Hertwig et al., 2004; Hertwig & Erev, 2009; Hertwig, 2012; Rakow, Demes, & Newell, 2008; Ungemach et al., 2009). The underweighting of rare events is a robust phenomenon: it is observed in the *obtained payoff* version of the task (in which the payoff of

the selected action becomes known to the participants), in the *foregone payoff* version of the task (in which the payoff of the non-selected action becomes known to the participants), and it is also observed in the *free-sampling* version of the task (where participants sample the actions as much as they want prior to the actual, single, decision) (Erev et al., 2010). By contrast, when choosing between explicitly-described lotteries participants tend to overweight rare events (Fox & Tversky, 1998; Kahneman & Tversky, 1979). The discrepancy in behavior between description-based and experience-based choices is known as the “*description-experience gap*” (Hertwig et al., 2004).

Sampling bias Hypothesis

It has been claimed (and hotly debated) that the underweighting of rare events, when learning from experience is due, at least in part, to a *sampling bias* owing to the reliance on a small sample (Barron & Erev, 2003; A. R. Camilleri & Newell, 2011; Denrell, 2007; Erev et al., 2010; Fox & Hadar, 2006; Hertwig et al., 2004; Hertwig, 2012; Hilbig & Glöckner, 2011; March, 1996) (but see (Barron & Ursino, 2013)). Sampling bias occurs in cases, in which a participant estimates the probability of a high reward (p), based on a finite sample of risky outcomes. The sample size, n , can correspond either to the total number of risky outcomes in the experiment, the size of a finite memory capacity (Hertwig, 2012; Kareev, Lieberman, & Lev, 1997), reflect a reliance on recent samples or a reliance on a sample from memory (see Definitions and Modeling Assumptions). Probability estimation that is based on a finite sample size may be inaccurate, because the actual number of risky trials with high outcome, denoted by k , may be different from its expectation value, $E(k) = n \cdot p$ (Hertwig et al., 2004). In the safe-risky schedule, this could bias participants’ choices for or against the risky alternative.

Objectives and Significance

In this paper we theoretically study the consequences of this sampling bias hypothesis and make the following three observations. First, we show that in fact, underweighting of rare events is not a natural consequence of the sampling bias hypothesis. Second, we show that in the safe-risky paradigm, the sampling bias makes a counter-intuitive prediction that minute changes in the parameters of the experiment are expected to lead to alternation between underweighting and overweighting of the rare events. To the best of our knowledge, there is no experimental support for this prediction and therefore, these two observations are a challenge to the sampling bias hypothesis. Finally, we show that robust underweighting of rare events is consistent with an alternative sequential learning model, assuming that initial conditions are reset.

Theoretical Analysis of the sampling bias Hypothesis

Definitions and Modeling Assumptions

Consider a participant in the safe-risky paradigm that contemplates between choosing the risky and safe alternatives after observing k high rewards out of n risky trials. As described above, one action in this paradigm, denoted as “*safe*,” is associated with a fixed payoff (M), whereas the other action, denoted as “*risky*,” is associated with a stochastic payoff that follows a Bernoulli distribution: a *high reward* (H) with a probability p and a *low reward* (L) with a probability $1 - p$. Undisclosed to the participants, the values of the parameters above (p, M, L, H) are chosen such that the expected returns associated with both alternative actions (safe and risky) are equal:

$$M = p \cdot H + (1 - p) \cdot L \tag{1}$$

For clarity, we first make three specific assumptions about the participant’s decision-making process, which will be later relaxed: (1) we assume that the participant’s estimate of the Bernoulli parameter (\hat{p}) is simply the empirical frequency of k high reward occurrences out of n considered risky trials $\hat{p} = k/n$. Such an estimate is known as the *maximum-*

likelihood estimator, ML (see Appendix). As noted above, the sample size, n , is not necessarily the total number of risky outcomes in the experiment. For example it can correspond to a sample from the memory of past outcomes (Erev et al., 2010); (2) we assume that the participant is risk neutral and (3) greedy. Assumptions (2) and (3) imply that the participant will choose the risky alternative if she believes that its expected reward is higher than that of the safe alternative.

Assumption (1), combined with the fact that in the safe-risky paradigm the expected returns are equal, imply that the participant will choose the risky alternative if $k/n > p$ and that she will choose the safe alternative if $k/n < p$. In case of indifference between the two alternatives ($k/n = p$) we assume that the participant chooses the two alternatives with equal probability. In fact, these three assumptions result in the previously-proposed *Sampler* model (Erev et al., 2010) for decision making.

To gain insight to the model, consider a simple case in which probability is estimated based on a single outcome ($n = 1$). In this case, there are only two possible outcomes: $k = 0$ with probability $(1 - p)$ and $k = 1$ with probability p . If the empirical frequency is the estimated probability (assumption 1), these correspond to an estimated probability of $\hat{p} = 0$ and $\hat{p} = 1$, respectively. Thus, underestimation of p is more likely than overestimation when $p < .5$ (and vice versa when $p > .5$), in line with the claim that sampling bias leads to underweighting of rare events.

Overweighting of Rare Events in the sampling bias Hypothesis

The extent to which the result described in the previous section for $n = 1$, namely, underestimation of p when $p < .5$ and overestimation of p when $p > .5$, holds for other sample sizes ($n > 1$) has been unclear. One study argued that sampling bias would result in underestimation of p when $p < .5$ and overestimation of p when $p > .5$ for any finite sample size n (March, 1996). Other studies have claimed that this intuition is only true when n is

“small”, without defining what this means quantitatively (Barron & Ursino, 2013; Denrell, 2007; Hau, Pleskac, & Hertwig, 2010; Hertwig et al., 2004; Hertwig, Barron, Weber, & Erev, 2006; March, 1996; Rakow et al., 2008).

In order to examine the generalization of the sampling bias hypothesis for $n > 1$, we first consider the particular case of $n = 2$. In this case, there are only three different possible outcomes, $k = 0, 1, 2$, and therefore only three values of estimated probabilities $\hat{p} = 0, .5, 1$, respectively. For rare events ($p < .5$) the probability of underestimating the probability is simply $\Pr(\hat{p} < p) = \Pr(\hat{p} = 0) = \Pr(k = 0) = (1 - p)^2$. Therefore, underweighting of rare events is more likely if and only if $(1 - p)^2 > .5$, which implies that $p < 1 - 1/\sqrt{2} \sim 0.3$.

Overestimation of the Bernoulli parameter is expected for $1 - 1/\sqrt{2} < p < .5$. These results demonstrate that in contrast to common lore (Hertwig et al., 2006; March, 1996), underestimation of the Bernoulli parameter is not expected for all $p < .5$ and sampling bias is consistent with both under and over estimation of the probability.

A standard measure of weighting of rare events in the safe-risky paradigm is a comparison between $\Pr(\text{risky})$ and $\Pr(\text{safe})$, estimated as the fraction of participants and/or trials in which the risky or safe alternative was chosen, respectively (Shafir et al., 2008). We consider the difference between the latter probabilities, denoted as the *probability estimation bias* (PEB), as a measure of the weighting of the probability:

$$\text{PEB} = \Delta\hat{p}(n, p) = \Pr(\text{risky}) - \Pr(\text{safe}) \quad (2)$$

A negative value of the PEB ($\Delta\hat{p} < 0$) indicates that the Bernoulli parameter is more likely to be underestimated than overestimated; a positive PEB ($\Delta\hat{p} > 0$) indicates that overestimation of p is more likely. A zero PEB ($\Delta\hat{p} = 0$) corresponds to a situation where participant are equally likely to overestimate and underestimate p .

With our assumptions (risky neutrality and greediness), $\Pr(\text{risky})$ is simply the sum of two terms: the probability that a sequence composed of n binary events will result in an estimation \hat{p} that is larger than p , $\Pr(\hat{p} > p | n, p)$ and half the probability of an unbiased estimate, $\frac{1}{2}\Pr(\hat{p} = p | n, p)$. Moreover, $\Pr(\text{safe}) = 1 - \Pr(\text{risky})$ thus the PEB is given by:

$$\Delta\hat{p}(n, p) = \Pr(\hat{p} > p | n, p) - \Pr(\hat{p} < p | n, p) \quad (3)$$

Assuming that the estimated parameter \hat{p} is simply the empirical frequency (assumption 1), the probabilities of overestimation and underestimation of p are:

$$\Pr(\hat{p} > p | n, p) = \sum_{np > k} \Pr(k | n, p) \quad (4)$$

$$\Pr(\hat{p} < p | n, p) = \sum_{np < k} \Pr(k | n, p) \quad (5)$$

where the probability of each value of k follows the binomial distribution:

$$\Pr(k | n, p) = \binom{n}{k} p^k (1-p)^{n-k} \quad (6)$$

The dependence of the PEB on p for the case of $n = 20$ is depicted in Figure 1 (top). Two points are noteworthy: first, the PEB is positive for many values of $p < .5$, indicating that for some values of p there is overweighting of rare events. Second, the dependence of the PEB on p is discontinuous (saw-like) with multiple zero-crossings, indicating that even a small change in the value of p may have a large effect on predicted behavior. For example, the value of PEB in a specific textbook example where $p = .1$ and $n = 20$ is negative ($-.07$) (Hertwig et al., 2006)¹, in line with underweighting of rare events (green circled dot in Figure 1, top). However, a minute decrease in the value of p to $p = .09$ results in a positive value of the PEB of .1 (blue circled dot in Figure 1, top)², predicting the opposite effect, the overweighting of rare events.

¹ $n = 20, p = .1$. $\Pr(\hat{p} > p) = \Pr(k > 2) = 0.32, \Pr(\hat{p} < p) = \Pr(k < 2) = .39$

² $n = 20, p = .09$. $\Pr(\hat{p} > p) = \Pr(k \geq 2) = 0.55, \Pr(\hat{p} < p) = \Pr(k < 2) = .45$

The dependence of the PEB on the sample size n is also saw-like, indicating that the expected behavior critically depends on the exact sample size. This is depicted in Figure 1 (bottom), where we plot the sign of the PEB for different pairs of p and n (black-underweighting; white - overweighting). The resultant striped pattern indicates that the value of the PEB changes its sign repeatedly as a function of both n and p , resulting in alternations between underestimation and overestimation of p . In the Appendix we prove that these non-continuous transitions occur (but not exclusively) when the product of n and p is an integer (thin vertical lines in Figure 1, top; green in Figure 1, bottom) and that increasing the value of p , starting from zero, the first transition from underestimation to overestimation of p occurs at $p = 1 - 2^{-\frac{1}{n}}$ (magenta in Figure 1, bottom), in line with the particular case of $n = 2$ put forward above. To the best of our knowledge there is no experimental evidence supporting this non-intuitive prediction, which is a challenge to the sampling bias hypothesis and the Sampler model.

The counterintuitive results of Figure 1 were based on three specific assumptions: maximum likelihood estimation of the probability, risk-neutrality and greediness. In the Appendix we mathematically show that overweighting of rare events emerges even when these assumptions are relaxed, for example when alternative Bayesian estimators are considered (e.g. mean of the posterior rather than the mode as in the Sampler model or assuming different priors), risk-sensitive utility function is used and exploration is introduced. Nevertheless, it should be noted that these different variations of the model may lead to different predictions (e.g., Bayesian estimation based on the mean of the posterior which minimizes the mean square error, with a uniform prior, overweighs rare events even for $n = 1$ in contrast to the maximum likelihood estimator). In other words, the bias in probability estimation is a function of both the experimental parameters p (and n) and the estimation strategy utilized by the participants.

Consequences for Experimental Design and Interpretation

The dependence of the PEB on p , depicted in Figure 1, may have important consequences when designing experiments and interpreting their results. For example, we consider the Technion choice prediction competition (CPC), in which a large dataset of choices in the safe-risky paradigm was collected for different problem sets, defined by the values of the reward schedule parameters (M , H , L , and p) (Erev et al., 2010). In the “sampling” condition, participants freely sampled the two alternatives before making a single decision. Overall, participants’ behavior in these experiments was consistent with underweighting of rare events and overweighting of common events, and the probability of choosing the risky alternative increased with p .

One of the models used to account for this behavior was the Sampler model (Erev et al., 2008), described above. Fitting this model to the empirical data, it was found that the value of n that is most consistent with the behavioral data (in the sense of least square error) is $n = 5$ (Erev et al., 2010), suggesting that humans use the Sampler strategy with $n = 5$. While this result may be interpreted as implying that in the brain, decisions are typically based on a memory of 5 samples, we argue, that this result namely the particular value of n extracted from the model, may actually reflect the particular choice of values of p used in the experiment. As shown in Figure 1, in general, the Sampler model predicts alternating underestimation and overestimation of p (Figure 2, blue line, for the particular case of $n = 5$). However, when considering only the 27 particular values of p used in the experiment, the Sampler model is consistent with the experimental findings, namely, underweighting of rare events, overweighting of common events, and increasing probability of choosing the risky alternative with p (Figure 2, blue markers). In fact, a straightforward calculation reveals that $n = 5$ is the only value of n in the range $1 < n < 100$ such that underweighting of rare events and monotonicity are respected in the Sampler model, for these particular values of p used in

the experiment. We hypothesize that if slightly different values of p would have been used in this experiment, robust underweighting would still have been observed but the resultant best-fit value of n in the Sampler model would have been very different.

It should be noted that a stochastic version of the Sampler model (aka *Stochastic Sampler*), in which the number of samples is a stochastic variable uniformly distributed between 1 and 9 that better fits the behavioral data (Erev et al., 2010) predicts underweighting of rare events for most values of p (Figure 2, red). The stochasticity of this model could represent heterogeneity between participants or variability in the sample size used by the single participant. Note that in the former case, the Sampler model predicts that a careful analysis of the choices made by single participant would reveal a non-monotonous dependence of estimated probability on p .

Sequential Learning as an Alternative Model

In the previous sections we considered the sampling bias hypothesis and demonstrated that the Sampler model does not account for the experimentally-reported robust underweighting of rare events. In this section we consider an alternative model, inspired by Reinforcement Learning models (Sutton & Barto, 1998), to account for the underweighting of rare events that is based on sequential learning: (e.g., Denrell, 2007; Shteingart, Neiman, & Loewenstein, 2013):

$$\hat{p}(t) = (1-\eta)\hat{p}(t-1) + \eta x(t) \quad (7)$$

Where $\hat{p}(t)$ and $\hat{p}(t-1)$ are the estimator at trials t and $t-1$, respectively, $\eta > 0$ is the learning rate and $x(t) \in \{0,1\}$ is the observation at trial t encoded such that the high reward H is 1 and the low reward L is 0. As a result, the estimated parameter \hat{p} is an exponentially weighted average of past observations.

We found that when the number of samples n is large compared with the reciprocal of the learning rate, then underweighting of rare events is expected for almost all³ values of p

and η . However, when the number of samples n is relatively small, the estimation of the probability strongly depends on the initial condition of Equation (7). For example, under the parsimonious assumption of symmetry ($\hat{p}(t=0) = .5$), overestimation of the probability is expected because of the residual effect of the initial condition⁴. This is demonstrated in Figure 3A for $\eta = 0.1$ and $n = 20$. However, a recent study has shown that in operant learning, outcome primacy is consistent with the resetting of initial conditions by first experience (Shteingart et al., 2013). Incorporating this reset in the probability estimation model of Equation (7), such that $\hat{p}(1) = x(1)$, reverses the sign of the probability estimation bias of Figure 3A and leads to robust underweighting of rare events, in line with experimental evidence. The reason is that as long as n is sufficiently small, the first event dominates the probability estimation, leading to underweighting of rare events as in the sampler model with $n = 1$. Extensive numerical simulations show that robust underweighting of rare events naturally emerges in the sequential learning model with reset of initial conditions for a large set of values of η and n (results not shown).

Discussion

There is a long tradition of experiments using the safe-risky paradigm. Behavior in these experiments is consistent with underweighting of rare events and overweighting of common events (e.g. Hertwig et al., 2004). It has been suggested that these biases result, at least in part, from a sampling bias of the Sampler model (e.g. Camilleri & Newell, 2011b). In this paper we challenge this sampling bias hypothesis. First, we show that sampling bias can also lead to overweighting of rare events, depending on the decision maker's characteristics (e.g., memory capacity, n). Second, we show that the sampling bias hypothesis predicts that

³ It has been previously argued (Denrell, 2007) that this is true for all values of p and η but an extensive numerical analysis (not shown) indicates that there are a few exceptions (e.g., $\eta = .45$ and $p = .45$).

⁴ Overestimation is guaranteed if $p < \hat{p}(t=0)(1-\eta)^n$, see Equation 7.

the magnitude, and even the sign of the probability estimation bias, critically depend on the exact value of p used in the experiment. The implication of this prediction is simple. One possibility, which we believe to be unlikely, is that careful experiments will reveal non-monotonic alternations between underweighting overweighting of rare events as a function of the underlying probability p , as predicted by the Sampler's sampling bias. Alternatively, if underweighting of rare events is indeed a robust phenomenon, then the sampling bias hypothesis, in the form of the Sampler model and its extensions (see Appendix), should be rejected.

We also proposed an alternative model for probability estimation that is based on sequential learning. When the value of the learning rate η is close to 1 or when the number of samples n is small, underweighting of rare events emerges because the estimation of the probability is based primarily on a single sample, the most recent sample in the former and the first sample in the latter. However, underweighting of rare events in that model is not restricted to these parameters and is observed for almost all values of η , n and p . The Stochastic Sampler model (with $n = 9$) is also consistent with the underweighting of rare events for almost all values of p and therefore can serve as an alternative account. These models are not necessarily exclusive as other models may also be consistent with robust underweighting of rare events. Additional research is required to characterize the family of models which are consistent with underwriting of rare events.

Experimental testing of the two (or other) models will require a careful analysis of sequential effects: first, the shape of the recency effect differs between the two models. Second and experimentally easier to identify, the sequential learning model predicts a transient primacy effect, which is not present in the Stochastic Sampler model but was observed in decision making from experience (Shteingart et al., 2013). Finally, the same cognitive strategies used to estimate probabilities in the safe-risky paradigm may also be

involved in other forms of probability estimation. Thus, the models may also be compared, perhaps more easily, by studying behavior in explicit probability estimation tasks in which the probabilities of two Bernoulli processes are compared.

Appendix

Mathematical Details

Maximum Likelihood Estimation

Given the likelihood $\Pr(k | n, p) = \binom{n}{k} p^k (1-p)^{n-k}$ the maximum likelihood estimator of p is defined as $\hat{p}_{ML} = \arg \max_p \Pr(k | n, p)$. Differentiating the likelihood and comparing it to zero results in $\hat{p}_{ML} = k/n$.

First transition (magenta line in Figure 1). Consider the case of $p < 1/n$. As long as $\Pr(k < np) = \Pr(k = 0) > .5$, the PEB is negative, resulting in underestimation of the probability. Because $\Pr(k = 0) = (1-p)^n$, negative PEB is guaranteed as long as $(1-p)^n > .5$ or $p < 1 - 2^{-\frac{1}{n}}$. Increasing p beyond the critical value of $p^* = 1 - 2^{-\frac{1}{n}}$ results in $\Pr(k = 0) < .5$. Note that $p^* = 1 - 2^{-\frac{1}{n}} < 1/n$ is consistent with the assumption that $p < 1/n$.

Discontinuity at integer np (green lines in Figure 1). If np is an integer, the median of the Binomial distribution (Equation 1) is equal to the mean np (Lord, 2010); thus, by the definition of the median, $P_+ + P_0 > .5$ and $P_- + P_0 > .5$, where $P_+ = \Pr(k > np | p)$, $P_- = \Pr(k < np | p)$, and $P_0 = \Pr(k = np | p)$. Consider the probability p' such that $p' = p + \varepsilon$ where $|\varepsilon| \ll \frac{1}{n}$. Because np' is not an integer, $P_0' = \Pr(k' = np' | p') = 0$. For $\varepsilon > 0$, $np' > np$ and, therefore, $P_- ' = \Pr(k' < np' | p') = \Pr(k \leq np | p) = P_- + P_0 + O(\varepsilon) > .5$,

resulting in a negative PEB. By contrast, for $\varepsilon < 0$, $np' < np$; thus $P_+' = \Pr(k' > np' | p') > .5$ and the PEB is positive.

Generalization to other Models of Cognitive Strategies

Safe indifference value generalization. We assume that the participant has an effective *safe-indifference value* (SIV) such that if the SIV is equal to the safe alternative reward (M), the participant is indifferent between the risky and safe alternatives and therefore chooses the risky alternative with probability .5. Moreover, we assume greediness, i.e., if the SIV is larger or smaller than M , the participant chooses the risky or the safe alternative, respectively. In general, the SIV is a function of k and n , $S(k, n)$. We further assume that (1) $S(k, n)$ monotonously increases with k , implying that the larger k , the more attractive is the risky alternative; (2) $S(k = 0, n) \geq L$ and $S(k = n, n) \leq H$, which implies that the risky alternative is not worse or better than receiving L or H with certainty, respectively. With these assumptions, we show that for $n \geq 2$, the PEB is a non-monotonic function of p .

To see that, note that in general, $S(k, n)$ attains one of $n+1$ possible values. Because $M = p \cdot H + (1-p) \cdot L$, there exists a probability p such that $S(0, n) < M < S(n, n)$. Without loss of generality, we assume that there exists a value k^* such that $S(k^*, n) < M < S(k^* + 1, n)$. In this framework, the probability that the risky alternative would be chosen is simply $\Pr(k > k^*)$. An increase in p has two consequences: first, it increases the probability of observing larger values of k . Thus, as long as k^* is fixed, it increases $\Pr(k > k^*)$. Second, it increases M due to the equal expected reward schedule of the safe-risky paradigm. Increasing the value of M beyond $S(k^* + 1, n)$, while keeping p fixed, will result in a discontinuous decrease in the probability of risky choice because the number of possible outcomes k that

would lead to a risky choice would decrease. Thus, increasing p will result in a continuous increase of the PEB (via p) interrupted by a discontinuous decrease of the PEB (via M), as depicted in Figure 1 (top).

The SIV is more general than the ML estimator above. It includes Bayesian prior or estimator other than ML. For example, an ML estimator with a beta-distribution prior

$$\Pr(p) \propto p(1-p) \quad \text{would result in } \hat{p} = \frac{k+1}{n+2}.$$

Note that in this case, the prediction is overestimation of the probability even for small p . For example, in the case of a single sample ($n=1$), $\hat{p} \geq \frac{1}{3}$. The latter estimator is also the *minimum mean square error estimator* (Stigler, 1986) with a flat uniform prior. In fact, it is well known in estimation theory that there is ambiguity between the prior distribution and the optimization criterion (Raiffa & Schlaifer, 1961). These and other Bayesian models will all lead to a SIV for every k and n . Moreover, SIV can incorporate subjective reward or risk sensitivity via a utility function u and a monotonously increasing probability weighting function w as in Prospect Theory.

Effect of exploration. According to the ε -greedy rule, the decision maker chooses the preferred alternative with probability $1-\varepsilon$ ($\varepsilon < 1$) and she uniformly chooses between the alternatives with probability ε . It is easy to see that the resultant PEB is simply scaled by $1-\varepsilon$ and therefore the lack of monotonicity and the zero-crossings are independent of the value of ε .

According to the softmax rule, $\Pr(\text{risky}) = 1/[1 + \exp(-\beta\Delta)]$, where β is a parameter and Δ is the difference in the “attractiveness” of the two alternatives. For concreteness we consider a risk-neutral participant, who estimates p using the ML estimator

$$\text{such that } \Delta = \frac{k}{n}(H-L) + L - M. \quad \text{It is easy to see that as long as } \beta \gg \frac{n}{H-L},$$

the graded nature of the softmax function does not have a substantial effect on the probability of choosing the risky alternative and therefore the PEB is non-monotonic. This intuition was

verified using numeric simulations that revealed that $\beta \approx \frac{3n}{H-L}$ is the critical value below which the PEB is non-monotonous (not shown).

References

- Ashourian, P., & Loewenstein, Y. (2011). Bayesian inference underlies the contraction bias in delayed comparison tasks. *PLoS ONE*, *6*(5), e19551.
doi:10.1371/journal.pone.0019551
- Barron, G., & Erev, I. (2003). Small feedback-based decisions and their limited correspondence to description-based decisions. *Journal of Behavioral Decision Making*, *16*(3), 215–233. doi:10.1002/bdm.443
- Barron, G., & Ursino, G. (2013). Underweighting rare events in experience based decisions: Beyond sample error. *Journal of Economic Psychology*, *39*, 278–286.
doi:10.1016/j.joep.2013.09.002
- Barron, G., & Yechiam, E. (2009). The coexistence of overestimation and underweighting of rare events and the contingent recency effect.
- Camilleri, A., & Newell, B. (2011). When and why rare events are underweighted: A direct comparison of the sampling, partial feedback, full feedback and description choice paradigms. *Psychonomic Bulletin & Review*, *18*(2), 377–384. doi:10.3758/s13423-010-0040-2
- Camilleri, A. R., & Newell, B. R. (2009). The role of representation in experience-based choice. *Judgment and Decision Making*, *4*(7), 518–529. Retrieved from <http://silk.library.umass.edu/login?url=http://search.ebscohost.com/login.aspx?direct=true&db=psyh&AN=2010-14354-001&site=ehost-live&scope=site>

- Camilleri, A. R., & Newell, B. R. (2011). Description- and experience-based choice: does equivalent information equal equivalent choice? *Acta Psychologica*, *136*, 276–284. doi:10.1016/j.actpsy.2010.11.007
- Denrell, J. (2007). Adaptive learning and risk taking. *Psychological Review*, *114*(1), 177–187. doi:10.1037/0033-295X.114.1.177
- Erev, I., Ert, E., Roth, A. E., Haruvy, E., Herzog, S. M., Hau, R., ... Lebiere, C. (2010). A Choice Prediction Competition : Choices from Experience and from Description. *Journal of Behavioral Decision Making*, *47*(1), 15–47. doi:10.1002/bdm
- Erev, I., Glozman, I., & Hertwig, R. (2008). What impacts the impact of rare events. *Journal of Risk and Uncertainty*, *36*(2), 153–177. doi:10.1007/s11166-008-9035-z
- Erev, I., & Haruvy, E. (2009). Learning and the economics of small decisions. In J. H. Kagel & A. E. Roth. (Eds.), *The handbook of experimental economics* (pp. 1–124). Retrieved from [http://www.econ.bgu.ac.il/seminars/monaster/attachments/February 5-2009.pdf](http://www.econ.bgu.ac.il/seminars/monaster/attachments/February%205-2009.pdf)
- Erlick, D. (1964). Absolute judgments of discrete quantities randomly distributed over time. *Journal of Experimental Psychology*, *67*(5), 475–482. doi:10.1037/h0042698
- Fox, C. R., & Hadar, L. (2006). “Decisions from experience” = sampling error + prospect theory. *Management*, *1*(2), 159–161.
- Fox, C. R., & Tversky, A. (1998). A belief-based account of decision under uncertainty. *Management Science*. Retrieved from <http://mansci.journal.informs.org/content/44/7/879.short>
- Friedman, D., & Massaro, D. W. (1998). The. *Psychonomic Bulletin, & Review*, *5*, 370–389.
- Gonzalez, R., & Wu, G. (1999). On the shape of the probability weighting function. *Cognitive Psychology*, *38*, 129–166. doi:10.1006/cogp.1998.0710
- Griffiths, T. L., & Tenenbaum, J. B. (2006). Optimal predictions in everyday cognition. *Psychological Science*, *17*(9), 767–73. doi:10.1111/j.1467-9280.2006.01780.x

- Hau, R., Pleskac, T. J., & Hertwig, R. (2010). Decisions From Experience and Statistical Probabilities : Why They Trigger Different Choices Than a Priori Probabilities. *Journal of Behavioral Decision Making*, 68(October 2009), 48–68. doi:10.1002/bdm
- Hau, R., Pleskac, T. J., Kiefer, J., & Hertwig, R. (2008). The Description – Experience Gap in Risky Choice : The Role of Sample Size and Experienced Probabilities. *Journal of Behavioral Decision Making*, 21(May), 493–518. doi:10.1002/bdm
- Hertwig, R. (2012). The psychology and rationality of decisions from experience. *Synthese*, 187(1), 269–292. doi:10.1007/s11229-011-0024-4
- Hertwig, R., Barron, G., Weber, E. U., & Erev, I. (2004). Decisions from experience and the effect of rare events in risky choice. *Psychological Science*, 15(8), 534–9. doi:10.1111/j.0956-7976.2004.00715.x
- Hertwig, R., Barron, G., Weber, E. U., & Erev, I. (2006). The role of information sampling in risky choice. In K. Fiedler & P. Juslin (Eds.), *Information sampling and adaptive cognition* (pp. 72–91). Cambridge University Press.
- Hertwig, R., & Erev, I. (2009). The description-experience gap in risky choice. *Trends in Cognitive Sciences*, 13(12), 517–523. doi:10.1016/j.tics.2009.09.004
- Hilbig, B., & Glöckner, A. (2011). Yes, they can! Appropriate weighting of small probabilities as a function of information acquisition. *Acta Psychologica*, 138, 390–6. doi:10.1016/j.actpsy.2011.09.005
- Kahneman, D., & Tversky, A. (1979). Prospect Theory: An Analysis of Decision under Risk. *Econometrica*, 47(2), 263–292. doi:10.2307/1914185
- Kareev, Y., Lieberman, I., & Lev, M. (1997). Through a narrow window: Sample size and the perception of correlation. *Journal of Experimental Psychology: General*, 126(3), 278–287. doi:10.1037/0096-3445.126.3.278

- Knill, D. C., & Richards, W. (1996). *Perception as Bayesian Inference*. (D. C. Knill & W. Richards, Eds.) (Eds.). (1996). *Perception as Bayesian inference*. Cambridge University Press. Cambridge University Press.
- Körding, K. (2007). Decision theory: what “should” the nervous system do? *Science (New York, N.Y.)*, 318(5850), 606–10. doi:10.1126/science.1142998
- Körding, K., & Wolpert, D. (2004). Bayesian integration in sensorimotor learning. *Nature*, 427(6971), 244–7. doi:10.1038/nature02169
- Kyburg, H. E. (1996). Probabilistic fallacies. *Behavioral and Brain Sciences*, 19(01), 31. doi:10.1017/S0140525X00041327
- Lord, N. (2010). Binomial averages when the mean is an integer. *Mathematical Gazette*, 94(530), 331–332.
- March, J. G. (1996). Learning to be risk averse. *Psychological Review*, 103(2), 309–319. Retrieved from <http://doi.apa.org/getdoi.cfm?doi=10.1037/0033-295X.103.2.309>
- Peterson, C. R., & Beach, L. R. (1967). Man as an intuitive statistician. *Psychological Bulletin*, 68(1), 29–46. Retrieved from <http://www.ncbi.nlm.nih.gov/pubmed/6046307>
- Pitz, G. F. (1965). Response variables in the estimation of relative frequency. *Perceptual and Motor Skills*, 21(3), 867–73.
- Raiffa, H., & Schlaifer, R. (1961). *Applied Statistical Decision Theory* (p. 354). Harvard University Press. Retrieved from <http://www.amazon.com/Applied-Statistical-Decision-Theory-Howard/dp/0875840175>
- Rakow, T., Demes, K. a, & Newell, B. R. (2008). Biased samples not mode of presentation: Re-examining the apparent underweighting of rare events in experience-based choice. *Organizational Behavior and Human Decision Processes*, 106(2), 168–179. doi:10.1016/j.obhdp.2008.02.001

- Raviv, O., Ahissar, M., & Loewenstein, Y. (2012). How recent history affects perception: the normative approach and its heuristic approximation. *PLoS Computational Biology*, 8(10), e1002731. doi:10.1371/journal.pcbi.1002731
- Shafir, S., Reich, T., Tsur, E., Erev, I., & Lotem, A. (2008). Perceptual accuracy and conflicting effects of certainty on risk-taking behaviour. *Nature*, 453(7197), 917–920. doi:10.1038/nature06841
- Shteingart, H., Neiman, T., & Loewenstein, Y. (2013). The role of first impression in operant learning. *Journal of Experimental Psychology. General*, 142(2), 476–88. doi:10.1037/a0029550
- Stigler, S. M. (1986). Laplace's 1774 Memoir on Inverse Probability. *Statistical Science*. doi:10.1214/ss/1177013620
- Sutton, R. S., & Barto, A. G. (1998). *Introduction to Reinforcement Learning*. Cambridge, MA, USA: MIT Press.
- Ungemach, C., Chater, N., & Stewart, N. (2009). Are probabilities overweighted or underweighted when rare outcomes are experienced (rarely)? *Psychological Science*, 20(4), 473–479. doi:10.1111/j.1467-9280.2009.02319.x
- Zhang, H., & Maloney, L. T. (2012). Ubiquitous log odds: a common representation of probability and frequency distortion in perception, action, and cognition. *Frontiers in Neuroscience*, 6(January), 1. doi:10.3389/fnins.2012.00001

Figures

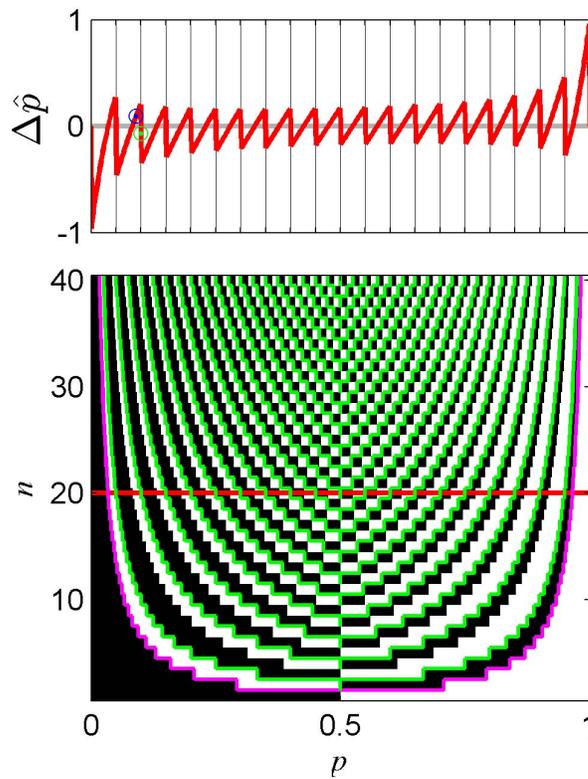


Figure 1. Top: Probability estimation bias (PEB, $\Delta \hat{p}$) for sample size $n=20$, as a function of the true probability p , using the maximum likelihood (ML) estimator. The thin grid of vertical lines are integer multiples of $1/n$ when $n = 20$. The PEB for the values of $p = .1$ and $p = .09$ are depicted by blue and green circled dots, respectively. Bottom: PEB sign (black and white for negative and positive, respectively) as a function of the sample size n (abscissa) and p (ordinate). The horizontal (red) line corresponds to $n=20$ as in the top panels. The magenta curve is the analytical bound of $p = 2^{-\frac{1}{n}}, 1 - 2^{-\frac{1}{n}}$. The green curves correspond to integer values of the product $n \cdot p$.

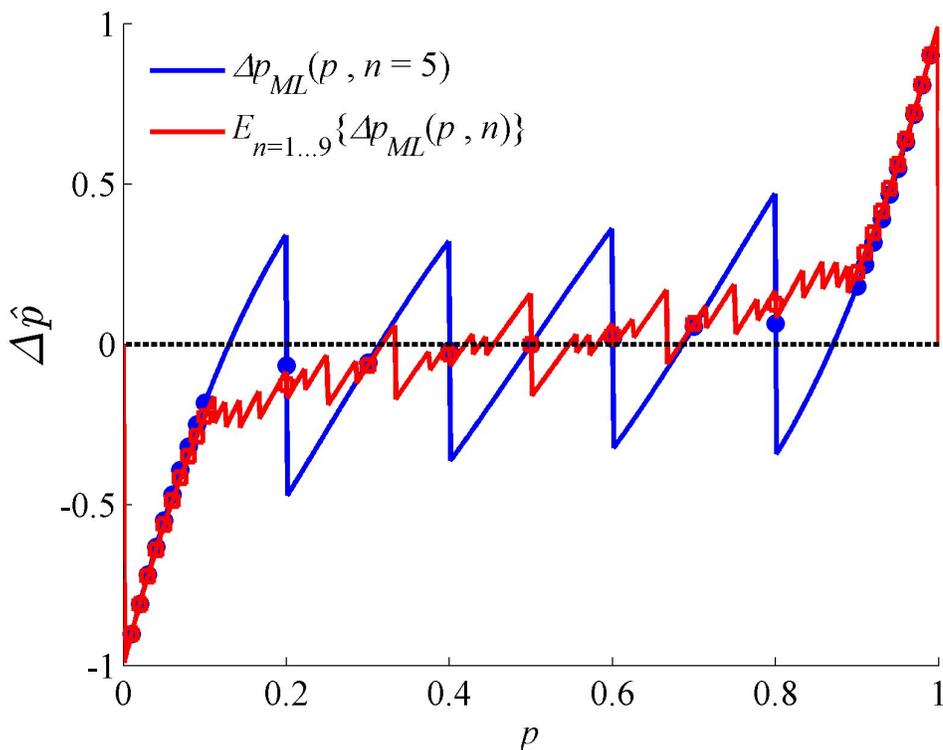


Figure 2. The probability estimation bias (PEB, y-axis) as a function of the underlying probability p (x-axis) for the Sampler with parameter $n = 5$ (blue) and Stochastic Sampler with $n = 9$ (red). Markers (blue circles and red squares) indicate the probability values used in the Technion Choice Prediction Competition (Erev et al., 2010). The figure demonstrates how the value of $n = 5$ avoids the non-monotonic nature of the PEB function as the values of the probabilities fall in integer multiples of $1/n$ for p larger than the one that guarantees underweighting (0.13). Moreover, the figure shows how even after introducing stochastic nature, the model still predicts overestimation of probabilities smaller than .5 (e.g., at probability $1/3$). Finally, notice how the two models are qualitatively different, yet align on the sampled probabilities (markers). The dashed horizontal black line represents a zero PEB.

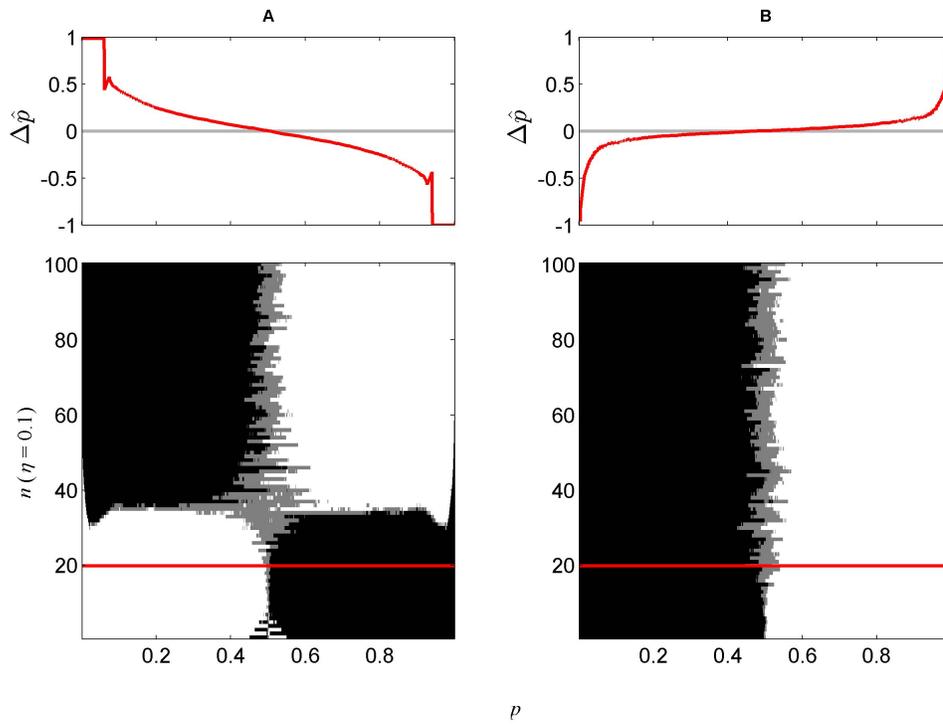


Figure 3. A, B (top), Probability estimation bias (PEB, $\Delta \hat{p}$) as a function p for a sequential learning model with $\eta = .1$ ($n = 20$). A, symmetric initial conditions and for reset of initial conditions. Simulation is based on 100,000 epochs each of which of 100 trials. Error bars are smaller than the width of the line. Bottom, sign of the PEB for all values of n between 1 and 100. Black and white areas correspond to significant underestimation and overestimation of p , respectively. Gray colors denote regions, in which the value of the PEB was not significantly different from 0 ($P > .05$).