האוניברסיטה העברית בירושלים

THE HEBREW UNIVERSITY OF JERUSALEM

NETWORKS OF RIGHTS IN CONFLICT: A TALMUDIC EXAMPLE

By

BARRY O'NEILL

Discussion Paper # 677 (December 2014)

מרכז פדרמן לחקר הרציונליות

THE FEDERMANN CENTER FOR
THE STUDY OF RATIONALITY

Feldman Building, Edmond J. Safra Campus, Givat-Ram,
Jerusalem 91904, Israel
PHONE: [972]-2-6584135 FAX: [972]-2-6513681
E-MAIL: ratio@math.huji.ac.il
URL: http://www.ratio.huji.ac.il/

Networks of Rights in Conflict: A Talmudic Example

December 24, 2014

Barry O'Neill

Department of Political Science
University of California, Los Angeles
barry.oneill@polisci.ucla.edu

Abstract

Many disputes involve conflicts of rights. A common view is that rights cannot really be in conflict so one of those being claimed must be a mistake. This idea leads to extreme outcomes that cut some parties out. Many studies have investigated how to choose a compromise among rights but they have focus on situations where the incompatibility comes from the *degrees* of the claims, as when, for example, a deceased person promised his heirs more than his total estate. I analyze a Talmudic problem where the difficulty is the *pattern* of the rights - each one trumps another in a cycle. The theory of non-transferable utility coalitional games suggests two solutions, one based on Shapley's and Maschler-Owen's values, which are equivalent for the problem, and the other on Harsanyi's and Kalai-Samet's, also equivalent. Each satisfies four out of five desirable properties, better than several other solutions. The NTU games are appropriate not just for power-based negotiation but for disputes over justice, fairness and rights. It is hoped that this analysis will form part of a general understanding of rights conflicts.

1. Introduction

Many disputes focus more on rights than on interests. For instance, two countries might claim the same territory, or someone might assert a right to perform an activity that another person takes as an infringement. A common view is that, unlike interests, rights must be compatible so one of the disputants must be wrong, and the question becomes which one. This attitude can lead to resolutions that cut some parties out completely. A slightly different approach also precludes compromises. The idea is that inconsistent rights can co-exist but one of them is so important that it overrides the other. When a local government takes someone's land to build a freeway, it compensates the owner but only at market value, as if the latter were selling it voluntarily. The right to choose what happens to one's property is set aside. The idea that one right fully trumps the other again leads to extreme resolutions.

The all-or-nothing philosophy follows from a narrow conception of what rights are. Nozick (1974) sees them as essentially negative, forbidding other parties, especially governments, from doing certain things to us – from restricting our speech, engaging in torture, and so on. Positive rights, like the right to property, an education or a nationality, are treated as pure rhetoric. If this understanding were true, rights conflicts could be avoided by the government refraining from taking any actions at all. However, as Waldron (1989) argues, even negative rights often call for positive actions. Our right not to be tortured requires the government to pass appropriate laws, protect us from torture by others, etc. The 1948 Universal Declaration of Human Rights and other international charters include positive rights and we would prefer a theory that gave a reasonable interpretation of them.

In certain Talmudic problems, parties have rights on a commodity that sum to more than the total available. A formal literature has arisen analyzing these (Aumann and Maschler 1985; O'Neill 1982). A man's marriage contracts assign his wives 300, 200 and 100 zuz, but he dies leaving only 200. The Talmud tells the court to divide the money 75, 75 and 50 without explaining why. Future scholars were left to puzzle over the reason but Aumann and Maschler have now apparently revealed it. The research has contributed to our understanding of fair division in general (Thomson 2013), and some of its findings have been brought back to the community of religious Talmudists (Aumann 1999). These *insufficient estate problems*, also called *bankruptcy* or *claims problems*, are relevant to the positive rights stated in international declarations: if everyone has the right to an education but the government lacks the money to satisfy them all, how should it divide what it has? This problem is structurally the same as the conflict among the widows.

The rights of the three widows are in conflict because of the *degrees* of their claims, but in another problem (*Ket.*, 95a)[1] the conflict comes from the *pattern* of who holds a right against whom. It can be called a *directed claims problem* since each party has claims only on certain others.[2] The story is this: A man owns a tract of land. His two wives will have a right to it if he dies and they need it to satisfy their marriage contracts. He wants to sell the land, so the first-married wife writes to the potential buyer forsaking her claim and the sale goes through. When the husband dies, the second wife takes the land from the buyer, arguing that her marriage contract predates his purchase and it was only the first wife, not her, who relinquished a claim. Then wife 1 takes the land from wife 2, pointing out that her marriage contract predates 2's and she herself gave up her claim only with respect to the buyer. The buyer takes it from wife 1 on the grounds of her letter, and so on, potentially forever, until they compromise.[3]

Insufficient estate and directed claims problems have a feature that forces us to figure out a compromise among the rights: the rights have the same source, e.g., all come from inheritance claims or all from contracts. If I claimed a right to cross my neighbor's private way to get my sick child to the hospital, and the neighbor said no, the inconsistent rights would have different bases. I could argue that preserving a life trumps his right in owning property. With the land problem, however, the rights are the same type so that there is no easy way to escape the conflict.

We wonder why the buyer did not insist on a promise from the second wife as well, but there might be no explanation for this. The Talmud often sets up unrealistic situations to illustrate its legal logic (Moskovitz 2002.) In the case of the three widows and the insufficient estate (*Ket.* 93a), each would normally take her due in order of her marriage date until the money was gone, so the sages devise a story that avoids the simple solution: no one knows their order because the husband married them all on the same day and none of the deeds recorded the hour.

[1] *Babylonian Talmud*, Steinsaltz edition, Vol. 12, Part IV, pp.153-154, *Ket.* 95a.

[2] Another problem combines structure and degree. Each party has his own account of who has which rights. Son 1 produces a will that divides the estate between 1 and 2, while son 2 has one dividing it between 2,3 and 4, etc. ("Sage of Jerusalem", 1887.)

[3] Naeh and Segal (2008) discuss other contexts producing cycles of rights.

Another oddity about the land problem is that the court does not settle it immediately. It lets the parties keep coming back with the same arguments. An explanation might be that the earlier insufficient estate problem was a single three-way dispute where each wife had rights against all the rest, so they went to court together. The present problem is really three two-way disputes. One holds the land and another claims it but the third person has no standing, and becomes involved only later when the court transfers the land to the second, in which case the first drops out. Only the parties themselves can solve it comprehensively. Our purpose is not exegesis of the Talmud, however, and we will view the land problem as a single dispute among the whole group, and look for a division by arbitration based on rights rather than one by negotiation based on power.

In contrast to the insufficient estate problem of *Ket.* 93, this mishnah does not say what the division should be. The answer seems obvious - the problem's symmetry suggests dividing the land equally. However adding another wife makes the problem asymmetrical (Figure 1,b) and solving this version requires a general theory. Here I consider several approaches, eventually focusing on two involving non-transferable utility (NTU) coalitional games.

The analysis is relevant to the game-theoretical issue of why there are so few applications of NTU methods. The world seems to offer many NTU-like situations but the literature has been abstract, treating mostly the solutions' general properties. Aumann (1985a, footnote 5) lists some applications but the reviews of McLean (2002) and Peters (2008) show that overall the literature is theory-oriented. The land problem lets us evaluate NTU solutions in a simple context and brings out some of their advantages.

Section 2 formalizes the idea of problems with directed claims. The most natural approach would use Markov chains but Section 3 shows it has disqualifying flaws. Section 4 translates the problem into an NTU game and calculates some examples. The idea is that a coalition can hold the land only if at least one of its members receives no outside claim; if there is such a member, the land stays with that group, but otherwise it goes to the complementary coalition where it is put to the same test. The game is NTU because side payments are seen as seizable just like the land itself, so the characteristic function is non-zero only for members receiving no outside claims. The solution concepts are Harsanyi's (1963), Shapley's (1967), Kalai and Samet's (1985), Maschler and Owen's (1992), as well as an NTU version of the nucleolus. Some coincide for these problems and in the end only three are numerically different. Section 5, the core of the paper, considers several persuasive requirements, either that certain payoffs be positive or that they respond monotonically to certain changes in parameters. Harsanyi's and Shapley's theories do the best.[4] Section 6 focuses on *tournaments*, where every pair has a claim between them in one direction or the other. Tournaments arise in problems like the Talmud's, that start with a full ranking - the timing of the marriages - then switch certain claims. Section 7 discusses two more approaches that fail in instructive ways, one using the sports rating systems originated by Zermelo, and the other based on a strategic bargaining model. Section 8 considers whether the NTU analysis can be seen as based on rights and justice rather than negotiating power.

2. Directed claims problems

We have in mind a group of people, one holding the land and each having a right to take it should particular others come to hold it. An *n-person directed claims problem* $C = (N, h, E)$ comprises a set $N = \{1, ..., n\}$, $n \geq 1$, of parties, an *initial holder* $h \in N$, and a set $E \in 2^{N \times N}$ of ordered pairs of parties, interpreted as the *claims.* Figure 1 indicates the initial holder by a circle, and the set of claims by a directed graph where "$i \to j$" means that if j possesses the commodity, i has a right to take it. The edge is referred to as"ij", so that $i \to j$ if and only if $ij \in E$. A condition is that $i \to j$ implies not $j \to i$, i.e., claiming is never mutual. It follows that $i \to i$ cannot hold. The set of claims problems is denoted $\mathcal{C}$.

For $C = (h, N, E)$ and π a permutation of N, let $\pi(E)$ be the permutation induced on E and $\pi(C)$ be the claims problem $(N, \pi(h), \pi(E))$. A function ϕ from $\mathcal{C}$ to R^n with i'th element ϕ^i is *anonymous* if $\phi^i(\pi(C)) = \phi^{\pi(i)}(C)$. A *solution function* ϕ is an anonymous function from $\mathcal{C}$ to R^{n+} with $\phi(C) = (x_1, \ldots, x_n)$ and $\sum x_i = 1$. It is interpreted as a proposal for a reasonable division of the land. Note that efficiency, anonymity and covariance with the size of the prize are built into the definition of a solution. We call $\phi(C)$ the *solution of* C *according to* ϕ.

The pair $G = (N, E)$ is C's *graph.* Following graph theory terminology, if for every distinct i, j either $i \to j$ or $j \to i$, the claims problem is a *tournament.* A tournament is *reducible* if there is a two-element partition $(S, N \backslash S)$ with $i \to j$ for every $(i, j) \in S \times N \backslash S$. In this case a reasonable solution function will grant everything to S, so a reducible tournament can be

[4] We name them after the author who devised them first.

treated as a smaller problem. The result of iteratively reducing a tournament is independent of the order of reduction.

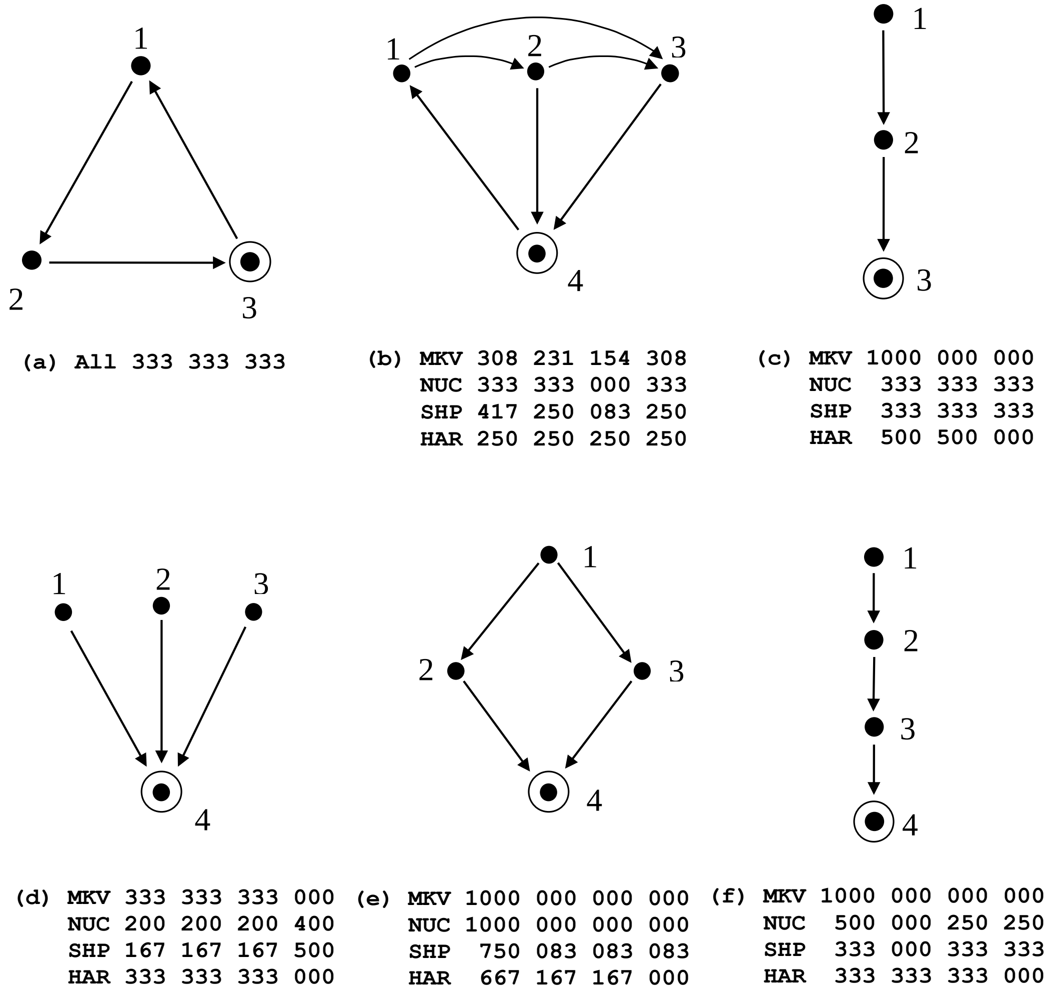

Figure 1. Some problems and their solutions. Initial holders are circled and decimals are omitted.

3. An unacceptable approach - the Markov solution

The Talmud's idea is that without a compromise the land will circulate forever, and this leads to a solution. At $t = 0$ the initial holder has the land, and at $t = 1, 2, 3\ldots$ we choose someone from those with a claim on the current holder equiprobably and independent of previous events, and give the land to that person. The *Markov solution* is the n-tuple of limiting relative frequencies for each one holding the land. That is, letting $X_{it} = 1$ if party i holds the land at t and $X_{it} = 0$ otherwise,

$$\phi^i_{MKV}(C) = \lim_{t\to\infty} E\left[\frac{\sum X_{it}}{t}\right].$$

The randomizing element follows the spirit of the "race to court" for insufficient estates (O'Neill 1982). Existence of the limits is guaranteed by the central limit theorem for Markov chains (Jackman 2009.) In the terminology of Markov chains a *communicating class* is a maximal set of parties such that for any ordered pair, if the first party holds the land at t, for some $t' > t$ the second will hold it with positive probability at t'. A *closed communicating class* is a set of states with the property that if the land starts in it, it is sure to stay in it. The Markov method gives 0 to parties outside a closed communicating class, and to those inside one it gives the likelihood of the chain entering the class times the limiting relative frequency of the particular party holding the land given the chain has entered it. Figure 1 shows its allocations for several cases.

The Markov solution, one could argue, removes incentives to renegotiate. After the n parties receive their shares, they could be seen as facing n new problems with each of them as the initial holder of whatever was just acquired, and with claims as in original graph. Solving these problems by the Markov method and summing each person's n allocations leaves their original totals unchanged. This fixed-point property holds for any solution function that is independent of reassigning the initial holder within the set of parties getting a positive share.

The Markov solution has a disqualifying flaw. Players who think strategically, as ours do, should reject it. In the 3-line of Figure 1(c) it would move the land from 3 to 2, then permanently to 1. But Wife 2 holds a compelling threat; she can refuse to seize the land from 3 unless 1 offers her a positive share. Her threat is credible since carrying it out would yield her no less than not doing that, and it is effective since it blocks any gain by 1. But according to the Markov solution she ignores her power and seizes the land knowing that she will lose it immediately. As to the attractiveness of the Markov's fixed-point property, the problem already assumes that the parties can make binding agreements so there is no point in immunizing a solution against renegotiation.

4. NTU methods

A coalitional approach has to specify whether utility is transferable (TU) among the players or not (NTU). Here the NTU assumption is more valid. A TU characteristic function for the Talmud's 3-cycle might have $v(i) = 0$ for single players, $v(ij) = 1$ for the pairs, and $v(123) = 1$. The interpretation is that a coalition can divide its worth v among its members in any way. However a legal system would probably let claimants seize the side payments as well as the original land. If coalition 12 forms and divides the benefit, Wife 2 is literally paying Wife 1 to not press a claim against her; this is equivalent to Wife 1 selling her claim to 2. The Buyer had a right against Wife 1 to the land so he should be able to go after 1's proceeds from the sale.[5] Accordingly 12's worth must stay entirely with 2, and this calls for an NTU approach.[6]

An *NTU game in coalitional form* is a pair (N, V) where N is a set of n players and V is a *characteristic function* from the non-empty subsets of N to the non-empty strict subsets of $\mathbf{R}^S$ that are closed, comprehensive and convex. The set $V(S)$ is interpreted as the utility vectors that S can achieve. In the Talmud's 3-cycle, player 1 can get at most 1 and player 2 can get at most 0 so $V(12) = \{(x_1, x_2) : x_1 \leq 1, x_2 \leq 0\}$. We will use abbreviations like $V(12) = (1, 0, -)$, where "-" is a place-holder to show which party is not in S. Similarly $V(123) = \{(x_1, x_2, x_3) : x_1 + x_2 + x_3 \leq 1\}$ is abbreviated (x_1, x_2, x_3).

4.1 Constructing an NTU characteristic function

The following is a reasonable way to define the characteristic function. Consider the 4-tournament of Figure 1(b). It is natural to set $V(34) = (-, -, 0, 0)$ since coalition 12 can seize the land from 34 whether 4 holds it or transfers it to 3. Regarding 13, the coalition could be seen as stronger than 34 since if 13 loses the land, it can immediately take it back. Still we treat the two coalitions alike on the premise that players must hold the land for

[5] We could think of the NTU version as having Wife 2 sell her claim to 1 for some negligible amount.

[6] Those curious about results from the TU Shapley value or nucleolus can refer to the values below, which for this problem coincide with the NTU values.

some time to derive any value from it. Even if the players in $N-S$ will not be able to keep the land, they will grab it to pressure S into concessions, so we set $V(13) = (0, -, 0, -)$.

This argument leads to the full characteristic function. For S containing the initial holder, the prize can be shared by those in S who are free of incoming claims. If everyone in S receives an in-claim, the prize goes to those in $N\backslash S$ free of in-claims. If there are none of the latter, everyone in the game gets 0. As mentioned, all the vectors are augmented by allowing free disposal.

Define K_S as the subset of S receiving no claims from $N-S$:
$K_S = \{i \in S : \nexists j \in N\backslash S, j \to i\}$

Then for $S \neq \emptyset, \subseteq N$,
$V(S) = \{x \in R^S : x_i \leq 0 \text{ for } i \in S\}$ if $K_S = \emptyset$, or if $h \in N\backslash S$ and $K_{N\backslash S} \neq \emptyset$;
$= \{x \in R^S : \sum x_i \leq 1 \text{ for } i \in K_S, x_i \leq 0 \text{ for } i \in S\backslash K_S\}$ otherwise.

It follows that $V(N) = \{(x_1, ..., x_n), \Sigma x_i \leq 1\}$.

Example 4.1. With the abbreviations, the characteristic function for the 3-person line, Figure 1(c), is
$V(1) = (0, -, -)$, $V(2) = (-, 0, -)$ $V(3) = (-, -, 0)$
$V(12) = (x_1, x_2, -)$, $V(23) = (-, 0, 1)$ $V(13) = (1, -, 0)$,
$V(123) = (x_1, x_2, x_3)$.

Example 4.2. For the 4-person diamond, Figure 1(e), it is
$V(1) = (0, -, -, -)$, $V(234) = (-, 0, 0, 1)$,
$V(2) = (-, 0, -, -)$, $V(134) = (x_1, -, x_3, 0)$,
$V(3) = (-, -, 0, -)$, $V(124) = (x_1, x_2, -, 0)$,
$V(4) = (-, -, -, 0)$, $V(123) = (x_1, x_2, x_3, -)$,
$V(12) = (x_1, x_2, -, -)$, $V(34) = (-, -, 0, 0)$,
$V(13) = (x_1, -, x_3, -)$, $V(24) = (-, 0, -, 0)$,
$V(14) = (1, -, -, 0)$, $V(23) = (-, 0, 0, -)$,
$V(1234) = (x_1, x_2, x_3, x_4)$.

4.2 NTU solution theories

Given this definition of the characteristic function what will the players receive? Noted solution theories are Harsanyi's (1963), Shapley's (1967), Maschler and Owen's (1992), and Kalai and Samet's (1985), as well as extensions of the TU nucleolus (e.g., Kalai 1975; Klauke 2002). Hart (2004) describes the first three and gives tips for calculating them. In all except Kalai and Samet's the solution depends on an n-vector of weights, interpretable as rates of utility transfer among the players. These weights do not reflect the players' real abilities to make side payments but are constructions arising from the particular game as steps to the solution. Generally NTU solutions admit different weight vectors yielding different allocations, but here we can focus on $(1, 1 \ldots, 1)$ since all the multiples of this vector give the same allocation, and, on account of the grand coalition's ability to achieve the unit simplex, unequal weights would unreasonably give some players zero (see Hart 2004 for a fuller explanation.)

Concerning a nucleolus-type NTU value, no single extension has become especially accepted but the existence of the external commodity of land again suggests 1:1 transfer rates. For an NTU characteristic function V, define the *corresponding TU characteristic function* $v : 2^N \to \{0, 1\}$ by setting $v(S)$ as the maximum sum of the allocations feasible for $V(S)$ based on the unit weight vector. The nucleolus used here is that of the corresponding TU game.

Some of the solutions concepts coincide for our games. Maschler and Owen's is the same as Shapley's because all coalitions have unit simplices as their Pareto-efficient sets. With the transfer weights taken as unit vectors, Kalai and Samet's agrees with Harsanyi's. The group produces only three numerically different solution functions, labeled ϕ_{NUC}, ϕ_{SHP} and ϕ_{HAR}. Figure 1 shows some results.

Example 4.3: Solutions for the 4-person diamond, Figure 1(e). For the nucleolus and Shapley value, V in Example 4.2 immediately gives the corresponding TU function v. The *minimal winning coalitions* (MWCs) are 12, 13 and 14, so ϕ_{NUC} is (1, 0, 0, 0), the unique allocation in the TU core, and ϕ_{SHP} is (3/4, 1/12, 1/12, 1/12). Harsanyi's value associates a *dividend* with each proper subset S of N. Each player receives a dividend from every such coalition he belongs to. A coalition's dividend is the same for all its members and the vector of sums of the dividends from S and its subsets must be in $V(S)$. This requires some dividends to be negative. This recursive process stops with coalitions of size $n-1$, and the values for the grand coalition are calculated by filling in the missing payoffs in the $(n-1)$-vectors to make them efficient, then averaging the n allocations. (See Hart 2004 for an example.) For the four person diamond the 1-person coalitions give their members 0. Based on the characteristic function in Example 4.2, the vectors of dividends from each 2-person coalition are

for 12, $(1/2, 1/2, -, -)$, for 13, $(1/2, -, 1/2, -)$,
for 14, $(0, -, -, 0)$, for 34, $(-, -, 0, 0)$,
for 24, $(-, 0, -, 0,)$ for 23, $(-, 0, 0, -)$.

For 3-person coalitions, 123 pays -1/3 and the rest pay 0. The accumulated dividends are then:
for 123, $(2/3, 1/6, 1/6, -)$, for 124, $(1/2, 1/2, -, 0)$,
for 134, $(1/2, -, 1/2, 0)$, for 234, $(-, 0, 0, 0)$.

Filling in the blanks to make each sum to 1, then averaging, $\phi_{HAR} = [(2/3, 1/6, 1/6, 0)+$

$$(1/2, 1/2, 0, 0) + (1/2, 0, 1/2, 0) + (1, 0, 0, 0)]/4$$
$$= (2/3, 1/6, 1/6, 0).$$

5. Payoff monotonicity and positivity

Certain changes in the game should help certain players and certain games should give particular players a positive payoff. This section considers five criteria. The nucleolus does poorly, the source of its problem being that it can be non-monotonic with the characteristic function (Megiddo 1974), even when it is limited to simple games (Montero, personal communication) like ours. None of the solutions satisfies all of the criteria but HAR and SHP achieve four out of five in their non-minimal versions.

Legal systems favor the person in physical possession of a property. The first criterion requires that if you become the initial holder your payoff does not fall. It has three versions of increasing strength.

Definition: A solution function ϕ satisfies *Minimal Initial Holder* iff for $C = (N, E, h)$ and $C' = (N, E, i), \phi^i(C) \leq \phi^i(C')$. It satisfies *Weak Initial Holder* iff the inequality is sometimes strict (for some player in some problem), and it satisfies *Strong Initial Holder* iff it is always strict.

Next, acquiring a claim should not hurt you:

Definition: A solution function ϕ satisfies *Minimal Out-claim* iff for $C = (N, E, h)$ and $C' = (N, E \cup \{i \to j\}, h)$, $\phi^i(C) \leq \phi^i(C')$. It satisfies *Weak Out-claim* iff the inequality is sometimes strict, and *Strong Out-claim* iff it is always strict.

Becoming free of an obligation should not hurt you:

Definition: A solution function ϕ satisfies *Minimal In-claim* iff for $C = (N, E, h)$ and $C' = (N, E\backslash\{j \to i\}, h), \phi^i(C) \leq \phi^i(C')$. It satisfies *Weak In-claim* iff the inequality is sometimes strict, and *Strong In-claim* iff the inequality is always strict.

According to the next lemma the characteristic function of Section 4 responds not at all or positively when the problem changes as above. Whether the solution function satisfies them then depends on whether it is weakly monotonic with the characteristic function.

Lemma 1: Suppose $C, C' \in \mathcal{C}$ are alike except that C' has initial holder i rather than h. Then $V_C(S) \subseteq V_{C'}(S)$ for all S with $i \in S \subseteq N$. Similarly when C and C' differ by i acquiring an out-claim or losing an in-claim.

Proof: By construction $V(S)$ depends only on $K_S, K_{N\backslash S}$ and whether $h \in S$. Also, $V(S)$ is (weakly) monotonic with K_S, anti-monotonic with $K_{N\backslash S}$ and monotonic with shifting the initial holder to a player in S. Someone in S acquiring an out-claim leaves the initial holder and K_S unchanged but possibly reduces $K_{N\backslash S}$. Someone in S losing an in-claim possibly increases K_S but leaves the other elements unchanged, and changing the initial holder leaves $K_S, K_{N\backslash S}$ unchanged. QED.

Acknowledgment, next, says that if j does better with i in the problem then i should also get something – i's contribution should be "acknowledged." For the sake of defining the game without i, i is assumed not to be the initial holder.

Definition: For $C = (N, E, h)$ and distinct $i \to j$ with $i \neq h$, let C_i be the problem generated by deleting i and its associated edges. A solution function ϕ satisfies *Weak Acknowledgment* iff $\phi^j(C) > \phi^j(C_i)$ implies $\phi^i(C) > 0$. It satisfies *Strong Acknowledgment* if, in addition, the condition sometimes holds, i.e., for some C, i and $j, \phi^j(C) > \phi^j(C_i)$.

Acknowledgment generalizes Section 3's argument against the Markov solution, which in the 3-line with 2 present gave everything to 1, but without 2 gave nothing to 1. Weak Acknowledgment holds for the rule of equal division, for example.

Weak Continuation requires that if party j gets a positive amount, so must everyone with a claim on j. It involves the core meaning of receiving a claim, that at least part of one's share should "continue on" to the claimant.

Definition: For $C = (N, E, h)$ a solution function ϕ satisfies *Weak Continuation* iff $\phi^j(C) > 0$ and $i \to j$ implies $\phi^i(C) > 0$. It satisfies *Strong Continuation* if $i \to j$ implies $\phi^i(C) > \phi^j(C)$.

Strong Continuation, that every claimant should get as much as or more than the claimee, is very demanding but HAR satisfies it (Theorem 1). It helps calculating that solution since if players are in a cycle they must receive the same payoff.

Example 5.1. In line with possession being "nine points of the law," the rule is this: assign fraction $a > 1/2$ to the initial holder h and divide the rest equally among those with claims on h. If there are none, divide the $1 - a$ equally among everyone but the initial holder. This rule satisfies Strong Initial Holder. It satisfies Weak Out-claim since a new out-claim helps a player if and only if the out-claim is on h. Minimal In-claim holds since eliminating an in-claim on someone does not alter that player's allocation. It satisfies Weak Acknowledgment since adding a new party never increases a current player's share. It violates Continuation, since those who are two steps from h with claims on those who are one step away get nothing. If we altered the rule by sharing the $1 - a$ among those with a directed path to h, it would satisfy Strong Acknowledgment and Weak Continuation.

Theorem 1.

(i) In their weakest versions the criteria are consistent, and no four imply the fifth.

(ii) The solution concepts satisfy the criteria as follows:

	MAR	*NUC*	*SHP*	*HAR*
Becoming the Initial Holder	weak	no	weak	weak
Losing an In-claim	no	no	weak	weak
Acquiring an Out-claim	weak	no	weak	weak
Acknowledgment	no	no	strong	no
Continuation	weak	no	no	strong

The Appendix gives the proof.

6. Tournaments

A tournament was defined as every pair of parties having exactly one claim between them. Problems generated by marriages or inheritances are often tournaments since they start with a complete order of the parties then reverse some of the claims, as the first wife did in her letter to the buyer.

Theorem 2: In a tournament MAR, NUC, SHP and HAR are invariant with respect to the initial holder.

Proof: For MAR: a tournament has exactly one closed communicating class since two or more would have arrows between them, contrary to their definition as being closed. Those outside the communicating class receive 0 and those inside receive amounts determined by the Markov chain central limit theorem, and these are independent of the initial holder.

For the NTU solutions: Section 4.1 defined K_S as the members of S with no claims from $N \backslash S$. If $K_S = \emptyset, V(S) = 0$. If $K_S \neq \emptyset$ and $K_{N \backslash S} = \emptyset$, $V(S)$ shares the prize among K_S. Both assertions are independent of the choice of h. The other situation $K_S, K_{N \backslash S} \neq \emptyset$ cannot arise in a tournament. QED.

Theorem 2 reduces the calculations for solutions of all tournaments of size n since we can ignore the choice of the initial holder. The numbers of distinct cases, unlabeled, appear in Table 1, column 1. They can be lowered further by considering only irreducible tournaments, whose numbers appear in column 2. These were calculated by combining the results of Moon (1968, Ch.6) and Wright (1970, formula 4). The unique irreducible 3-tournament is the Talmud's problem, and the unique 4-tournament is Figure 1(b). There only the first wife forsook her claim but if the second wife did so as well that would produce a permutation of that graph. Figure 2 shows the six irreducible 5-tournaments.

n	all	irreducible
1	1	1
2	1	0
3	2	1
4	4	1
5	12	6
6	56	35
7	456	353
8	6880	6008
9	191536	178133
10	9733056	9355949

Table 1: Numbers of non-isomorphic tournaments

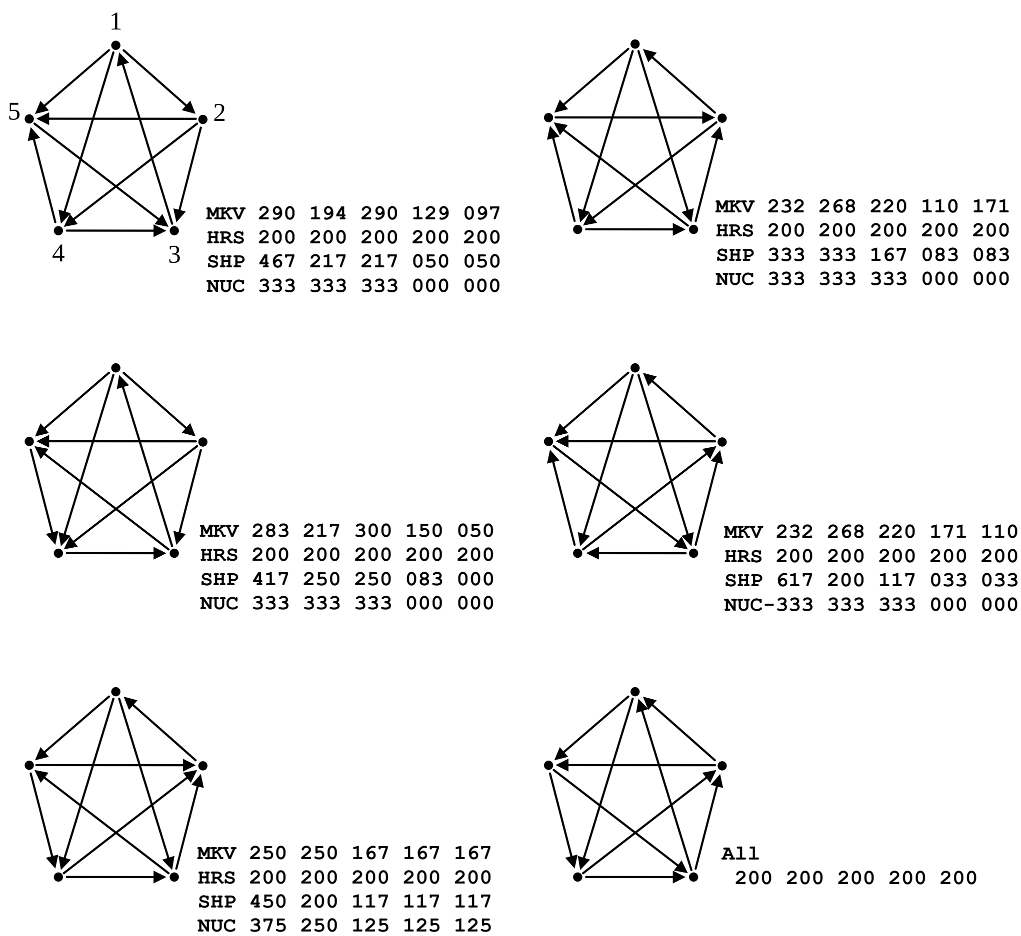

Figure 2: The six irreducible 5-tournaments.

Theorem 3. For any claims problem, if $K_S = \emptyset$ for all $S \neq \emptyset$, HAR divides the prize equally.

Proof: A subcoalition allocates the same dividend to all its members, so if $K_S = \emptyset$ for some $S \neq \emptyset$, then every $i \in S$ receives a zero dividend and the players' gains come only from their membership in N. QED.

Corollary. In an irreducible tournament HAR divides the prize equally.

7. Two further unacceptable methods - sports scoring and strategic games

7.1. Zermelo's scoring system

In chess, rugby, tennis and some other games, competitors go against each other pairwise, generating a matrix with the numbers of wins, ties and losses. Systems have been developed to assign a rank or score to each competitor overall. Having a claim on someone in the land problem is analogous to beating them in a sports match, so the scores calculated by one of the sports systems could be taken as the legal strengths of their cases in order to determine the division of the land. Claims problems have every pair of players "compete" with each other once, so they are a special case of the sports tournaments.

The first author to develop a rigorous method was Zermelo (1929). In a simple version certain pairs of players meet in matches, perhaps multiple times, which one or the other wins. The outcome depends probabilistically on their unobserved strengths: the likelihood of i beating j is $s_i/(s_i + s_j)$. This chance element can give rise to cycles. Assuming wins are independent conditional on the strengths, the likelihood of a particular matrix is proportional to

$$\prod_{i \to j \, \in \, E} \frac{s_i}{s_i + s_j}$$

.

The *Zermelo solution* $\phi_{ZRM}(C)$ is the vector $\boldsymbol{s}$ that maximizes the above likelihood, normalized by $\Sigma s_i = 1$. The estimated strengths depend, it turns out, entirely on the vector of wins and not on whom the wins are against. They are more widely spaced than the wins, in the sense that the ratio of two different strengths is greater than that of the corresponding numbers of wins. The 4-tournament of Figure 1(b) has win vector (2, 2, 1, 1) and Zermelo allocations (375, 375, 125, 125).

Unfortunately the method is hard to justify in the context of rights. One party's claim for the land is not a random event but a continuing fact and the court would choose the same pairwise winner every time. The sports analogy would be each competitor having an Achilles' heel that certain others could always exploit, but the scoring method's assumption about strengths determine the winner does not fit such situations and it would do a poor job of predicting the winners.

Another difficulty is that sometimes Zermelo's equations have no interior solution. This arises when some ordered pair of players has no directed path connecting them. For the three-line Zermelo gives (1, 0, 0), and so makes no prediction for a match between 2 and 3. Conner and Grant (2000) and others have extended the method to such cases but their approaches have arbitrary elements. This flaw plus the lack of justification in the rights context lead us to not consider it here.

7.2. A strategic approach

The land problem can be interpreted as a strategic game. Suppose that i is the holder at time t. If no one has a claim on i, then i keeps the land. Otherwise, choose j from the claimants on i, equiprobably and independently of past events. Player j proposes a division, while threatening either to Seize the land from i or to Not Seize it. Mixed strategy threats are also possible. At the next stage each of the others will Accept or Refuse the proposed division, and j's threat is conditional, assigning an action Seize or Not Seize to each pattern of their responses. Then each of the others simultaneously Accept or Refuse j's proposal. If all Accept, it becomes permanent but if someone Refuses, j carries out the threat and the procedure is repeated at $t+1$ with the new holder, either i or j, as the proposer. Players maximize their long-term average payoffs.

Randomly assigning the power to propose is a typical way of translating a coalitional game into a strategic one, but here it gives unreasonable allocations and there is no clear way to fix it. All the perfect equilibria for the 3-line give all the land to Player 2. Player 2 can propose (0, 1, 0) and threaten to Seize if 3 is the lone refuser, Not Seize if 1 is the lone refuser, and Seize with some intermediate probability if both refuse. Player 2 is presenting 1 and 3 with a 2x2 sub-game where each has Accept as a weakly dominant strategy. However their using it seems less than sensible for a reason that is informative for the strategic approach generally. Players 1 and 3 have common knowledge that they have nothing to gain and something to lose by from choosing Accept, but 1 decides that she might as well do so since otherwise the prize will stay with 3, while player 3 reasons the same way. The unilateral-deviation logic of the equilibrium precludes them from mutually expecting their joint refusal, although either one can only gain or break even from that. To avoid this problem, some equilibrium in the game must be coalition-proof (Bernheim, Peleg and Whinston 1987; Moreno and Wooders 1996), but there are none of these .

The usual transformation of a coalitional to a strategic game gives one random player all the initiative (Serrano 2005). The others do not interact, not even to form shared expectations of how to promote their joint interest. The coalitional approach avoids this problem because all players simultaneously consider the benefits of the coalitions they might join. Every subset can coordinate to improve its payoffs. Persistence might lead to a better strategic model but the coalitional solutions above respond directly to the problem.

8. NTU solutions as appropriate for rights conflicts

What makes the NTU approaches especially suited to rights conflicts? They are often seen as representing power-driven negotiations but here we want to support them from principles of justice and fairness. The difference between the two rationales is important since the point of endowing people with rights is to shield them from others' power.

A power-based rationale might start by viewing an NTU game as an incomplete representation of a strategic game. It fills in a dynamic mechanism, a subgame of what would happen if the parties did not agree to the solution's division and acted on their own. The nucleolus, for example, would be seen as the result of different coalitions threatening to block proposed allocations, with the credibility of each one's threat measured by how much a coalition would lose by refusing to go along with the proposal. The nucleolus aggregates the losses of the coalitions into an order and minimizes them. However the concept can also be justified on fairness and equity by regarding the coalitions as equalizing their sacrifices compared to what they have a "right" to receive.

Fairness and justice rationales are typically expressed through axioms rather than dynamic mechanisms. Examples include paying for a public good or allotting legislature seats based on districts' populations (Thomson 2001). Most of the axioms fall under one or more of four "meta-principles," related to fairness and justice. One group states which factors are *relevant* to the outcome, and a second states which are *irrelevant* to it. A third says that holding the right should be *beneficial* to the holder, not be neutral or harmful, and a fourth group says that the problem should be solved in a way that is *efficient* the whole group.

For example, the best-known Talmudic division problem (*Baba Mezi'a*, 2a) has one party claiming that a garment is "all mine" while the other says "half of it is mine." The court divides it 3/4, 1/4. The judgment presumably follows from several premises: the whole garment gets divided (the efficiency meta-principle); the uncontested half goes entirely to the first party (holding a right is a benefit), while the other half is split equally (relevance of their claims and irrelevance of everything else). Finally, each party gets the sum of the two operations (irrelevance of the return from one subproblem to that from the other.) Similarly Nash's axioms for two-person bargaining solution can also be interpreted as describing fairness in arbitration and identified with the four meta-principles. The outcome must be based on the feasible set and the disagreement point (relevance and irrelevance), be better than what each could get acting alone (benefit), be Pareto-efficient (efficiency), and be independent of utility transformations and non-chosen alternatives (irrelevance.)

For NTU games fairness and justice rationales are found in axiomatic theories, e.g., Hart (1985) and de Clippel *et al.* (2004) for Harsanyi's solution, Aumann (1985b) for Shapley's, and Hart (2005) for Maschler and Owen's, among others, whose axioms are readily classifiable into the four categories. A type of axiom that is typically central involves the idea that sums of games are to be solved additively,"sum" and "additive" being defined in different ways. As in the two-hold-a-garment problem, additivity is essentially the meta-principle of irrelevance. The point is that the axiomatizations of NTU theories show that they can be interpreted as expressing parties' rights.

A counterargument would point out that the characteristic function V used here comes from strategic considerations. Coalitions hypothetically take the land from each other or shift it back and forth internally to prevent the opposing coalition from taking it. However, the involvement of strategic moves is compatible with the theory being one of rights. Many appeals to rights havee strategic elements. Some rights become active only when the possessor claims them - philosophers of law call them *privileges* or *liberties.*

Someone questioned by the police has the right to remain silent or to talk; someone with a stock option has the right to exercise it or not. Whether to assert their rights is their calculated decision.

9. Conclusion

The Talmud's land problem presents one variety of rights conflict and the NTU approaches provide a reasonable solution. This result augments past work on bankruptcy problems. A question for future research is what other cases of incompatible rights can be made simple, precise and amenable to formal analysis. The program would be to build up our understanding of such conflicts brick by brick.

Acknowledgments: The author thanks Javier Arin, Robert Aumann, Maria Montero, Catherine Ross, David Schmeidler, Muhamet Yildiz and especially Dov Samet for helpful suggestions and discussions. I am grateful for support from the Center for the Study of Rationality at the Hebrew University, the Leon Levy Foundation, the Institute for Advanced Study, Princeton, and the Center for Advanced Study in the Behavioral Sciences at Stanford. Comments are very welcome.

APPENDIX

Proof of Theorem 1

(i) The rule of equal division satisfies all the criteria in their weakest forms.

To show that no four criteria imply the fifth: The table indicates that this holds for Continuation and Acknowledgment. For Initial Holder, divide the amount equally among everyone but the initial holder.

To show the independence of Out-claim, consider the following rule: First, each i receives weight $1/(n+1)$, then divides it equally among those others with no claim on i. (If i receives an in-claim from everyone else, the weight goes to them equally.) Then the $1/n+1$ not allocated is divided equally among everyone, and the prize is divided in proportion to the total weight each receives. This rule satisfies Initial Holder trivially, and satisfies Acknowledgment and Continuation weakly because all players get a positive share. It satisfies In-claim, because deleting $i \to j$ leaves j's share constant, but it violates Out-claim.

For In-claim, reverse the above rule - have each i divide $1/(n+1)$ among those whom i has a claim on, etc.

(ii) The proof will refer to these pairs of problems:
Pair A: $(\{1,2\}, \emptyset, 2)$ versus $(\{1,2\}, \{12\}, 2)$. (All four rules give $(0,1)$ and $(1,0)$ respectively.)
Pair B: the 3-line, Figure 1(c), versus the 3-cycle, Figure 1(a).
Pair C: $(\{1,2,3\}, \{13,12,23\}, 3)$ versus the 3-line. (HAR gives respectively $(1,0,0)$ and $(1/2,1/2,0)$.)
Pair D: $(\{1,2\}, \{12\}, 1)$ versus $(\{1,2\}, \emptyset, 1\}$.

MAR satisfies Weak Initial Holder: If the new initial holder i is in the same closed communicating class as the original, then i's payoff is unchanged. If not, then in the original game i got 0. The criterion holds weakly: becoming the initial holder may help (two players holding no claims) or make no difference (a 3-cycle.)

NUC violates Initial Holder: With claims {12, 15, 24, 25, 31, 32, 43, 54} and initial holder 1 the MWCs are $\{\overline{13}, \overline{125}, \overline{234}, \overline{345}\}$ and the nucleolus is (1/5, 1/5, 2/5, 0, 1/5). Shifting the initial holder to 3 changes the MWCs to

$\{\overline{13}, \overline{34}, \overline{1245}\}$ giving nucleolus (1/3, 0, 1/3, 1/3, 0), lowering 3's payoff. [7]

SHP and HAR satisfy Weak Initial Holder, Weak In-claim and Weak Out-claim: Extending the TU-context definitions of Young (1985) and Shubik (1962), an NTU solution ϕ satisfies *coalitional monotonicity* iff for any games V and W on the same player set, $V(T) \subset W(T)$ for some T and $V(S) = W(S)$ for all $S \neq T$ imply $\phi^i(S) \leq \phi^i(W)$ for all $i \in T$. Applying this condition repeatedly shows it is equivalent to the condition that $V(S) \subset W(S)$ for all $S \ni i$ and $v(S) = w(S)$ for all S not containing i imply $\phi^i(V) \leq \phi^i(W)$. As indicated by the Lemma, the changes in the three criteria maintain or enlarge the payoff sets of player i for all coalitions containing that player. The formula for the TU Shapley value shows that it is coalitionally monotonic, and the present ϕ_{SHP} is identical to it. The Kalai-Samet solution also satisfies coalitional monotonicity (Kalai and Samet 1985) and coincides with ϕ_{HAR} for claims problem games.

Each criterion is satisfied weakly. Regarding Initial Holder, this is shown for SHP and HAR using the examples for MAR, above. CHECK : Regarding In-Claim, by SHP adding one may hurt the recipient (2 in Pair A above) or make no difference (1 in Pair B). By HAR it may hurt (2 in Pair A) or make no difference (3 in Pair C). Regarding Out-Claim, by SHP a new out-claim may help (1 in Pair A) or make no difference (3 in Pair B). By HAR it may help (1 in Pair A) or make no difference (1 in Pair D).

MAR violates In-claim: In {12, 23, 34, 41, 31}, a 4-person cycle augmented with a diagonal, the Markov solution is (2/7, 2/7, 2/7, 1/7), independent of the initial holder. Removing the diagonal 31 leads to solution (1/4, 1/4, 1/4, 1/4), worse for party 1. The intuition is that dropping 3's claim on 1 removes a shortcut that brought the land back to 1 sooner.

NUC violates In-claim: In the 5-player problem with claims {21, 24, 25, 31, 32, 43, 53, 54} and initial holder 1 the MWCs are $\{\overline{23}, \overline{25}, \overline{345}\}$ and the nucleolus is (0, 1/3, 1/3, 0, 1/3). Removing claim 53 yields MWCs $\{\overline{23}, \overline{125}, \overline{134}, \overline{245}, \overline{345}\}$ and nucleolus (1/7, 2/7, 2/7, 1/7, 1/7), lowering 3's payoff.

MAR satisfies Weak Out-claim: Consider two graphs G and H containing

[7] The lemma suggests that the nucleolus' negative result arises from its violation of coalitional monotonicity (below). The author is grateful to Maria Montero for an interesting counterexample within the class of simple games, and her game is worth including here as it is more transparent towards its point than the one in the proof or than Megiddo's counterexample. To MWCs $\{\overline{123}, \overline{124}, \overline{125}, \overline{134}, \overline{135}, \overline{234}, \overline{235}\}$ add $\overline{145}$. The nucleolus goes from (1/3, 1/3, 1/3, 0, 0) to (1/5, 1/5, 1/5, 1/5, 1/5), harming player 1.

players i and j, differing only in that in H, $i \to j$. Let m and $m+1$ be the respective in-degrees of j. (Note that moving from a claims problem to the graph of the corresponding Markov chains reverses the arrows, so in the Markov graphs these become the out-degrees.) We assume $m \geq 1$ and that i and j get positive allocations in G, otherwise the assertion is trivial. The goal is to show that i gets at least as much in H. The allocation to i is the inverse of the duration between the chain's starting at i and first returning there. The only part of that expectation affected by the new link is the expected first passage duration from j to i, which will be shown to be the same or lower. For G, let p be the probability that the process, starting at j, reaches j before i, and let T_1 be the expected duration for this event, given that it happens. Since state i is recurrent, $p < 1$. The corresponding values for H are $\frac{m}{m+1}\ p$ and T_1. For G the event that the process, starting at j, reaches i before j has probability $1-p$. Let the expected duration for this event given it happens be T_2. For H these values are $\frac{1}{m+1} + p\frac{m}{m+1}$ and $\frac{1}{m+1} + (1-p)\frac{m}{m+1}$ T_2. Letting T_G and T_H be the mean first passage times from j to i,

$$\begin{aligned} T_G \ &= p(T_G + T_1) + (1-p)T_2 \\ &= \ \frac{p}{1-p}\ T_1 \ + \ T_2. \end{aligned}$$

Similarly,

$$\begin{aligned} T_H &= \frac{pm}{m+1}[T_1 + T_H] + \frac{1}{1+m} + \frac{(1-p)m}{m+1}\ T_2 \\ &= \frac{p}{1-p+1/m}T_1 + \frac{1+(1-p)m\ T_2}{1+(1-p)m}. \end{aligned}$$

In the corresponding two final expressions, the coefficient of T_1 in the expression for T_H is less than that for T_G, and the second term for T_H lies in $[1, T_2)$. Thus the new link increases i's allocation if $T_1 > 0$. but never decreases it. Finally, MAR satisfies the criterion weakly because a new Out-claim may help (1 in Pair A) or make no difference (1 in Pair C).

NUC violates Minimal Out-claim: The example of NUC violating Initial Holder had nucleolus (1/5, 1/5, 2/5, 0, 1/5). Adding claim 35 yields MWCs $\{\overline{13}, \overline{34}, \overline{1245}\}$ and nucleolus (1/3, 0, 1/3, 1/3, 0), decreasing 3's payoff.

MAR violates Acknowledgment: In the 3-line of Figure 1(c), party 1 gets a positive payoff and 2 gets 0, but in $(\{1,3\}, \emptyset, 3)$, the same problem without 2, 1 gets nothing.

NUC violates Acknowledgment: Claims {14, 15, 21, 25, 32, 42, 43, 53, 54} and initial holder 1 yields MWCs $\{\overline{12}, \overline{145}, \overline{234}\}$ and nucleolus (1/3, 1/3, 0, 1/3, 0). Eliminating 5 gives claims {14, 21, 32, 42, 43}, MWCs $\{\overline{12}, \overline{13}\}$ and nucleolus (1, 0, 0, 0). Player 4 needs 5 to get a positive payoff, nevertheless 5 gets nothing.

SHP satisfies Weak Acknowledgment: Assume the opposite, that $\phi^j_{SHP}(C) > \phi^j_{SHP}(C_i)$ for $j \neq i$ but $\phi^i_{SHP}(C) = 0$. Then i is a dummy in the corresponding TU game: $v_C(S) = v_C(S \cup i)$, so $\phi^j_{SHP}(C) = \phi^j_{SHP}(C_i)$. Also, 1's payoffs in $\{\{1,3\}, \emptyset, 3\}$ and the 3-line show that SHP satisfies it non-trivially.

HAR violates Acknowledgment: With the graph {12, 24, 43, 13} and initial holder 3, Harsanyi allocates (1/2, 1/2, 0, 0). Removing 4 gives graph {12, 13}, initial holder 3, and solution (1, 0, 0), so 4 is necessary for 2's payoff but gets nothing.

MAR satisfies Weak Continuation: If i is one of k parties with a claim on j, then $\phi^i_{MAR} \geq (1/k)\phi^j_{MAR} > 0$. Since the inequality is strict MAR satisfies Continuation at least weakly. Figure 1(b) shows it does not satisfy it strongly.

SHP violates Continuation: Figure 1(f).

HAR satisfies Strong Continuation: Supposing $i \to j$, player j receives payoffs from two sources, each of which gives as much or more to i. For one, any $S \subset N$ such that $i, j \in S$ gives i and j equal non-negative total dividends. For $S \subset N$ with $j \in S$ and $i \notin S$, i's claim on j blocks any gain by j. Possibly j receives a further $1/n$ at the final stage, but this source will yield more to j than i, (will yield, respectively, $1/n$ and 0), only if (1) no $S \subseteq N \backslash \{j\}$ has $V(S)$ a unit simplex of full-dimension (dimension $|S| - 1$), and (2) some $S \subseteq N \backslash \{i\}$ has $V(S)$ of full dimension. These conditions are incompatible: (1) implies that any S satisfying (2) contains j, so $i \in N - S$, therefore V(S) limits j to 0 and $V(S)$ is less-than-full-dimensional. It follows that $\phi^i_{HAR}(C) \geq \phi^j_{HAR}(C)$.

References

Aumann, R., 1985a. Notes and comments on the non-transferable utility value: a comment of the Roth-Shafer examples. Econometrica 53, 667-677.

Aumann, R., 1985b. An axiomatization of the non-transferable utility value. Econometrica 53, 599-612.

Aumann, R., 1999. On the man with three wives [Hebrew] Moriah 22, 98-107.

Aumann, R., Maschler, M., 1985. Game-theoretic analysis of a bankruptcy problem from the Talmud. J. Econ. Theory 36, 195-213.

Bernheim, D., Peleg, B., Whinston, M., 1987. Coalition-proof Nash equilibria, I, concepts. J. Econ. Theory 42, 1-12.

Conner, G. R., Grant, C.P., 2000. An extension of Zermelo's model for ranking by paired comparisons. Eur. J. Appl. Math. 11, 225-247.

de Clippel, G., Peters, H., Zank, H., 2004. Axiomatizing the Harsanyi solution, the symmetric egalitarian solution and the consistent Shapley solution for NTU-games. Int. J. Game Theory 33, 145-158.

Harsanyi, J., 1963. A simplified bargaining model for the n-person cooperative game. Int. Econ. Rev. 4, 194-220.

Hart, S., 1985. An axiomatization of Harsanyi's nontransferable utility solution. Econometrica 53, 1295-1313.

Hart, S., 2004. A comparison of non-transferable utility values. Theory and Decision 56, 35-46.

Hart, S., 2005. An axiomatization of the consistent non-transferable utility value. Int. J. Game Theory 33, 355-366.

Jackman, S., 2009. Bayesian Analysis in the Social Sciences. Wiley.

Kalai, E., 1975. Excess functions for cooperative games without sidepayments. SIAM J. App. Math. 29, 60-71.

Kalai, E., Samet, D., 1985. Monotonic solutions for cooperative games. Econometrica 53, 307-327.

Klauke, S., 2002. NTU Pre-Nucleoli. Doctoral Dissertation, University of Bielefeld.

Maschler, M., Owen, G., 1992. The consistent Shapley value for games without side payments. In: Selten, R. (Ed.), Rational Interaction. Springer-Verlag, pp. 4-12.

McLean, R., 2002. Values of non-transferable utility games. In: Aumann, R., Hart, S. (Eds.), Handbook of Game Theory III, North Holland, pp. 2077-2120.

Megiddo, N., 1974. On the non-monotonicity of the bargaining set, the kernel and the nucleolus of a game. SIAM J. App. Math. 27, 355-358.

Moon, J.W., 1968. Topics on Tournaments. Holt, Rinehart and Winston.

Moreno, D., Wooders, J., 1996. Coalition-proof equilibrium. Games Econ. Beh. 17, 82112.

Moscovitz, L., 2002. Talmudic Reasoning: From Casuistics to Conceptualization. Mohr Siebeck Verlag.

Naeh, S., Segal, U., 2008. The Talmud on Transitivity. Working Paper, Talmud Department, Hebrew University.

Nozick, R., 1974. Anarchy, State and Utopia. Blackwell.

O'Neill, B., 1982. A problem of rights arbitration from the Talmud. Math. Soc. Sci. 2, 345-371.

Peters, H., 2008. NTU-games. In: Derigs, U. (Ed.), Optimization and Operations Research. EOLSS Publishers.

"Sage of Jerusalem" 1887. Article 12. [Hebrew] Torah Mizion. 1, 2.

Serrano, R., 2005. Fifty years of the Nash program, 1953 - 2003. Investigaciones Economicas 29, 219-258.

Shapley, L., 1967. Utility Comparison and the Theory of Games. Paper P3582, RAND Corporation. Available at: http://www.rand.org/pubs/papers/P3582/

Shubik, M., 1962. Incentives, decentralized control and the assignment of joint costs and internal pricing. Man. Sci. 8, 325-343.

Thomson, W., 2001. On the axiomatic method and its recent applications to game theory and resource allocation. Soc. Choice Welfare. 18, 327-386.

Thomson, W., 2013. Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: an update. Rochester Center for Economic Research Working Paper. Available at: http://rcer.econ.rochester.edu/RCERPAPERS/rcer_578.pdf

Waldron, J., 1989. Rights in conflict. Ethics 99, 503-519.

Wright, E.M., 1970. The number of irreducible tournaments. Glasgow Math. J. 11, 97-101.

Young, H.P., 1985. Monotonic solutions of cooperative games. Int. J. Game Theory 14, 65-72.

Zermelo, E., 1929. Die Berechnung dur Turnier-Ergebnisse als ein Maximumproblem der Wahrscheinlichkeitsrechnung. Math. Zeit, 29, 436-260. Translated in part in H. A. David, A.W.F. Edwards, eds., 2001, Annotated Readings in the History of Statistics. Springer-Verlag.